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Cubics Related to Coaxial Circles

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Abstract. This note generalizes a result of Paul Yiu on a locus associated with a triad of coaxial circles. We present an interesting family of cubics with many properties similar to those of pivotal cubics. It is also an opportunity to show how different ways of writing the equation of a cubic lead to various geometric properties of the curve.

1. Introduction

In his Hyacinthos message [7], Paul Yiu encountered the cubic **K360** as the locus of point P (in the plane of a given triangle ABC) with cevian triangle XYZ such that the three circles $AA'X$, $BB'Y$, $CC'Z$ are coaxial. Here $A'B'C'$ is the circumcevian triangle of X_{56}, the external center of similitude of the circumcircle and incircle. See Figure 1. It is natural to study the coaxiality of the circles when $A'B'C'$ is the circumcevian triangle of a given point Q.

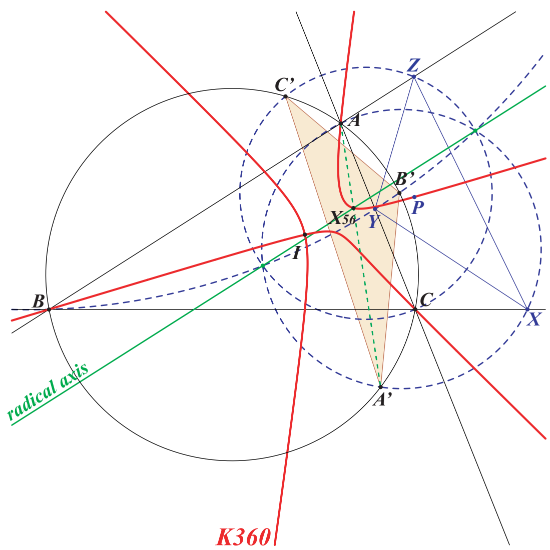

Figure 1. $\mathbf{K360}$ and coaxial circles

Throughout this note, we work with homogeneous barycentric coordinates with reference to triangle ABC, and adopt the following notations:

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gX the isogonal conjugate of X
tX the isotomic conjugate of X
cX the complement of X
aX the anticomplement of X
tg$X$ the isotomic conjugate of the isogonal conjugate of X

2. Preliminaries

Let $Q = p : q : r$ be a fixed point with circumcevian triangle $A'B'C'$ and P a variable point with cevian triangle $P_aP_bP_c$. Denote by $\mathcal{C}_A$ the circumcircle of triangle $AA'P_a$ and define $\mathcal{C}_B, \mathcal{C}_C$ in the same way.

Lemma 1. *The radical center of the circles $\mathcal{C}_A, \mathcal{C}_B, \mathcal{C}_C$ is the point Q.*

Proof. The radical center of the circumcircle $\mathcal{C}$ of triangle ABC and $\mathcal{C}_B, \mathcal{C}_C$ must be Q. Indeed, it must be the intersection of BB' (the radical axis of $\mathcal{C}$ and $\mathcal{C}_B$) and CC' (the radical axis of $\mathcal{C}$ and $\mathcal{C}_C$). Hence the radical axis of $\mathcal{C}_B, \mathcal{C}_C$ contains Q. □

These three radical axes are in general distinct lines. For some choices of P, however, these circles are coaxial. For example, if $P = Q$, then the three circles degenerate into the cevian lines of Q and we regard these as infinite circles with radical axis the line at infinity. Another trivial case is when P is one of the vertices A, B, C, since two circles coincide with $\mathcal{C}$ and the third circle is not defined.

Lemma 2. *Let H be the orthocenter of triangle ABC. For any point $Q \neq H$ and $P = H$, the circles $\mathcal{C}_A, \mathcal{C}_B, \mathcal{C}_C$ are coaxial with radical axis HQ.*

Proof. When $P = H$, the cevian triangle of P is the orthic triangle $H_aH_bH_c$. The inversion with respect to the polar circle swaps A, B, C and H_a, H_b, H_c respectively. Hence the products of signed distances $HA \cdot HH_a$, $HB \cdot HH_b$, $HC \cdot HH_c$ are equal but, since they represent the power of H with respect to the circles $\mathcal{C}_A, \mathcal{C}_B, \mathcal{C}_C$, H must be on their radical axes which turns out to be the line HQ. If $Q = H$, the property is a simple consequence of the lemma above. □

3. The cubic $\mathcal{K}(Q)$ and its construction

Theorem 3. *In general, the locus of P for which the circles $\mathcal{C}_A, \mathcal{C}_B, \mathcal{C}_C$ are coaxial is a circumcubic $\mathcal{K}(Q)$ passing through H, Q and several other remarkable points. This cubic is tangent at A, B, C to the symmedians of triangle ABC.*

This is obtained through direct and easy calculation. It is sufficient to write that the radical circle of $\mathcal{C}_A$, $\mathcal{C}_B$, $\mathcal{C}_C$ degenerates into the line at infinity and another line which is obviously the common radical axis of the circles. This calculation gives several equivalent forms of the barycentric equation of $\mathcal{K}(Q)$. In §§4 – 9 below, we explore these various forms, deriving essential geometric properties and identifying interesting points of the cubic. For now we examine the simplest of all these:

$$\sum_{\text{cyclic}} b^2c^2\, p\, x\,(y+z)(ry-qz) = 0 \iff \sum_{\text{cyclic}} \frac{x(y+z)}{a^2}\left(\frac{y}{q}-\frac{z}{r}\right) = 0. \quad (1)$$

It is clear that $\mathcal{K}(Q)$ contains A, B, C, Q and the vertices A_1, B_1, C_1 of the cevian triangle of $\mathbf{tg}Q = \frac{p}{a^2} : \frac{q}{b^2} : \frac{r}{c^2}$. Indeed, when we take $x = 0$ in equation (1) we obtain $(b^2ry - c^2qz)yz = 0$.

$\mathcal{K}(Q)$ also contains $\mathbf{ag}Q$. Indeed, if we write $\mathbf{ag}Q = u : v : w$ then $v+w = \dfrac{a^2}{p}$, etc, since this is the complement of $\mathbf{ag}Q$ i.e. $\mathbf{g}Q$. The second form of equation (1) obviously gives $\displaystyle\sum_{\text{cyclic}} \frac{u}{p}\left(\frac{v}{q} - \frac{w}{r}\right) = 0$.

Finally, it is easy to verify that $\mathcal{K}(Q)$ is tangent at A, B, C to the symmedians of triangle ABC. Indeed, when b^2z is replaced by c^2y in (1), the polynomial factorizes by y^2.

3.1. *Construction.* Given Q, denote by S be the second intersection of the Euler line with the rectangular circumhyperbola $\mathcal{H}_Q$ through Q.

Let $\mathcal{H}'_Q$ be the rectangular hyperbola passing through O, Q, S and with asymptotes parallel to those of $\mathcal{H}_Q$.

A variable line L_Q through Q meets $\mathcal{H}'_Q$ at a point Q'.

L_Q meets the rectangular circumhyperbola through $\mathbf{g}Q'$ (the isogonal transform of the line OQ') at two points M, M' of $\mathcal{K}(Q)$ collinear with Q.

Note that Q is the coresidual of A, B, C, H in $\mathcal{K}(Q)$ and that $\mathbf{ag}Q$ is the coresidual of A, B, C, Q in $\mathcal{K}(Q)$. Thus, the line through $\mathbf{ag}Q$ and M meets again the circumconic through Q and M at another point on $\mathcal{K}(Q)$.

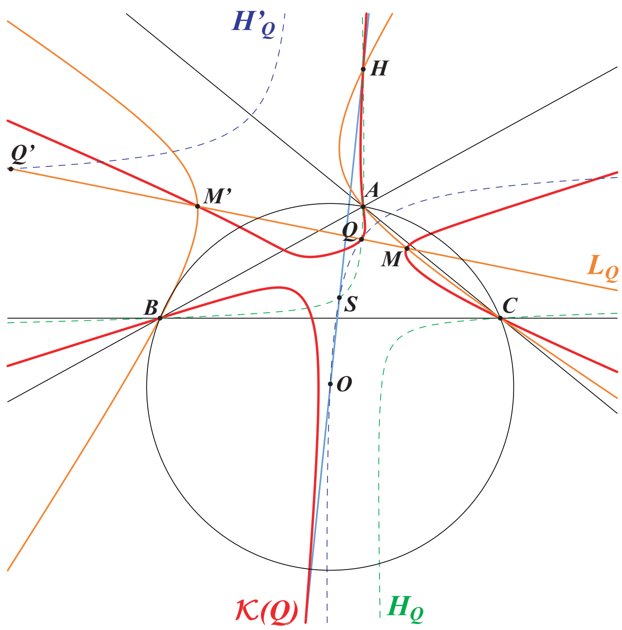

Figure 2. Construction of $\mathcal{K}(Q)$

4. Intersections with the circumcircle and the pivotal isogonal cubic $\mathrm{p}\mathcal{K}_{\mathrm{circ}}(Q)$

Proposition 4. *$\mathcal{K}(Q)$ intersects the circumcircle at the same points as the pivotal isogonal cubic $\mathrm{p}\mathcal{K}_{\mathrm{circ}}(Q)$ with pivot $\mathbf{ag}Q$.*

Proof. The equation of $\mathcal{K}(Q)$ can be written in the form

$$\begin{aligned}&\sum_{\text{cyclic}}(-a^2qr+b^2rp+c^2pq)\,x\,(c^2y^2-b^2z^2)\\&+(a^2yz+b^2zx+c^2xy)\sum_{\text{cyclic}}p\,(c^2q-b^2r)\,x=0.\end{aligned}\tag{2}$$

Any point common to $\mathcal{K}(Q)$ and the circumcircle also lies on the cubic

$$\sum_{\text{cyclic}}(-a^2qr+b^2pr+c^2pq)\,x\,(c^2y^2-b^2z^2)=0,\tag{3}$$

which is the pivotal isogonal circumcubic $\mathrm{p}\mathcal{K}_{\mathrm{circ}}(Q)$. □

The two cubics $\mathcal{K}(Q)$ and $\mathrm{p}\mathcal{K}_{\mathrm{circ}}(Q)$ must have three other common points on the line passing through G and $\mathbf{ag}Q$. One of them is $\mathbf{ag}Q$ and the two other points E_1, E_2 are not always real points. Indeed, the equation of this line is

$$\sum_{\text{cyclic}}p(c^2q-b^2r)x=0.$$

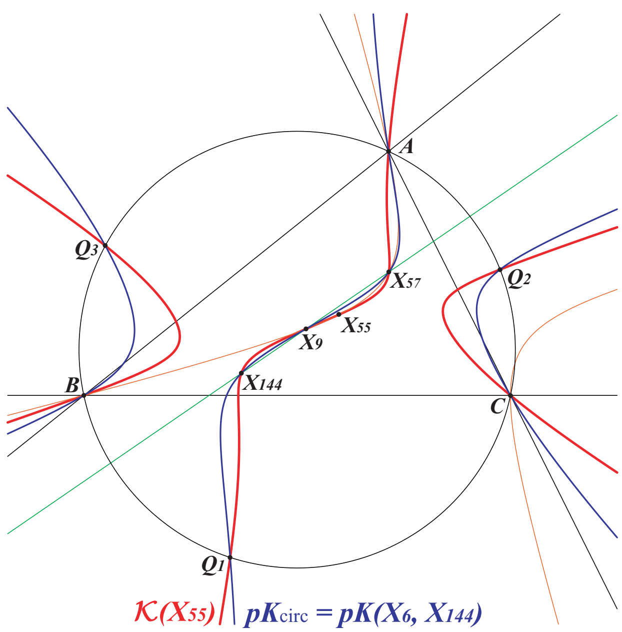

Figure 3. $\mathcal{K}(Q)$ and $\mathrm{p}\mathcal{K}_{\mathrm{circ}}(Q)$ when $Q=X_{55}$

These points E_1, E_2 are the intersections of the line passing through G, $\mathbf{g}Q$, $\mathbf{ag}Q$ with the circumconic $ABCKQ$ which is its isogonal conjugate. It follows that these points are the last common points of $\mathcal{K}(Q)$ and the Thomson cubic **K002**.

Figure 3 shows these cubics when $Q = X_{55}$, the isogonal conjugate of the Gergonne point X_7. Here, the points E_1, E_2 are X_9, X_{57} and $\mathbf{ag}Q$ is X_{144}.

Thus, $\mathcal{K}(Q)$ meets the circumcircle at A, B, C with concurrent tangents at K and three other points Q_1, Q_2, Q_3 (one of them is always real). Following [4], **ag**Q must be the orthocenter of triangle $Q_1Q_2Q_3$.

4.1. *Construction of the points* Q_1, Q_2, Q_3. The construction of these points again follows a construction of [4] : the rectangular hyperbola having the same asymptotic directions as those of $ABCHQ$ and passing through Q, $\mathbf{ag}Q$, the antipode Z on the circumcircle of the isogonal conjugate Z' of the infinite point of the line $O\mathbf{g}Q$ meets the circumcircle at Z and Q_1, Q_2, Q_3. Note that Z' is the fourth point of $ABCHQ$ on the circumcircle. The sixth common point of the hyperbola and $\mathcal{K}(Q)$ is the second intersection Q' of the line $H\mathbf{ag}Q$ with both hyperbolas. It is the tangential of Q in $\mathcal{K}(Q)$. It is also the second intersection of the line ZZ' with both hyperbolas. See Figure 4.

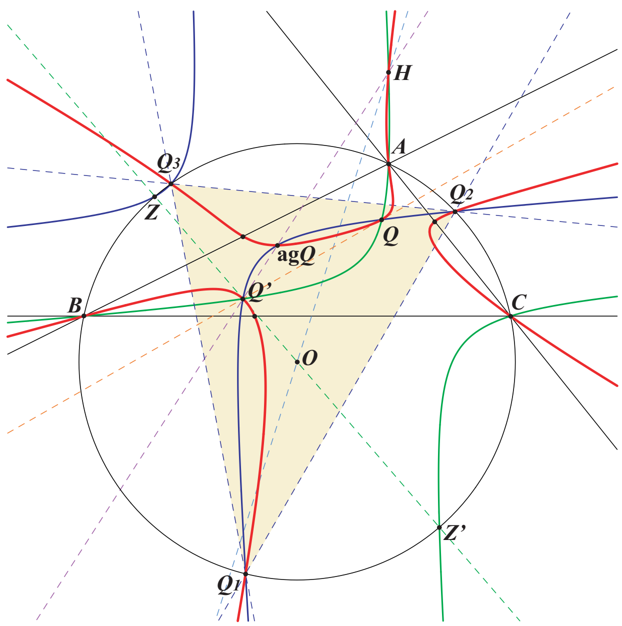

Figure 4. Construction of the points Q_1, Q_2, Q_3

These points Q_1, Q_2, Q_3 have several properties related with Simson lines obtained by manipulation of third degree polynomials. They derive from classical properties of triples of points on the circumcircle of ABC having concurring Simson lines.

Theorem 5. *The points Q_1, Q_2, Q_3 are the antipodes on the circumcircle of the three points Q'_1, Q'_2, Q'_3 whose Simson lines pass through* $\mathbf{g}Q$.

It follows that Q_1, Q_2, Q_3 are three real distinct points if and only if $\mathbf{g}Q$ lies inside the Steiner deltoid $\mathcal{H}_3$.

Theorem 6. *The Simson lines of Q_1, Q_2, Q_3 are tangent to the inconic $\mathcal{I}(Q)$ with perspector $\mathbf{tg}Q$ and center $\mathbf{cg}Q$. They form a triangle $S_1S_2S_3$ perspective at $\mathbf{cg}Q$ to $Q_1Q_2Q_3$.*

S_1 is the common point of the Simson lines of Q'_1, Q_2, Q_3. These points S_1, S_2, S_3 are the reflections of Q_1, Q_2, Q_3 in $\mathbf{cg}Q$. See Figure 5.

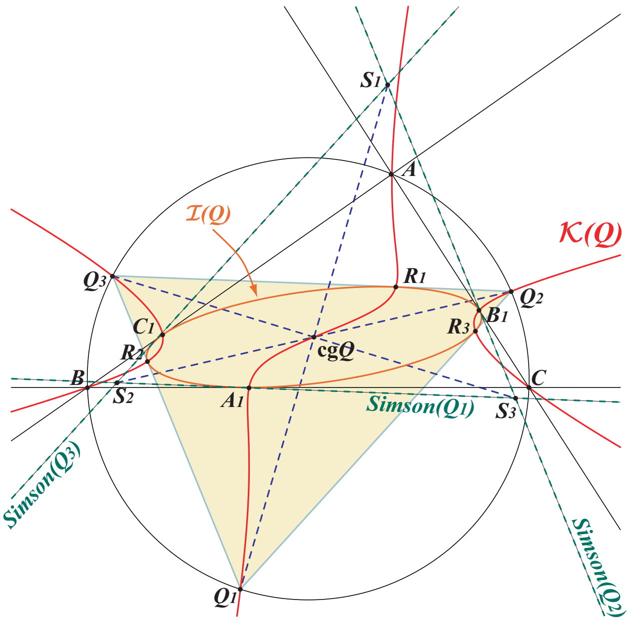

Figure 5. $\mathcal{K}(Q)$ and Simson lines

Another computation involving symmetric functions of the roots of a third degree polynomial gives

Theorem 7. *$\mathcal{K}(Q)$ meets the circumcircle at A, B, C with tangents concurring at the Lemoine point K of ABC and three other points Q_1, Q_2, Q_3 where the tangents are also concurrent at the Lemoine point of $Q_1Q_2Q_3$.*

This generalizes the property already encountered in a family of pivotal cubics seen in [4, §4]. Since the two triangles ABC and $Q_1Q_2Q_3$ are inscribed in the circumcircle, there must be a conic inscribed in both triangles. This gives

Theorem 8. *The inconic $\mathcal{I}(Q)$ with perspector $\mathbf{tg}Q$ is inscribed in the two triangles ABC and $Q_1Q_2Q_3$. It is also inscribed in the triangle formed by the Simson lines of Q_1, Q_2, Q_3.*

$\mathcal{K}(Q)$ meets $\mathcal{I}(Q)$ at six points which are the contacts of $\mathcal{I}(Q)$ with the sidelines of the two triangles. Three of them are the vertices A_1, B_1, C_1 of the cevian triangle of $\mathbf{tg}Q$ in ABC. The other points R_1, R_2, R_3 are the intersections of the sidelines of $Q_1Q_2Q_3$ with the cevian lines of H in $S_1S_2S_3$. In other words, $R_1 = HS_1 \cap Q_2Q_3$, etc. See Figure 5. Note that the reflections of R_1, R_2, R_3 in the center **cg**Q of $\mathcal{I}(Q)$ are the contacts T_1, T_2, T_3 of the Simson lines of Q_1, Q_2, Q_3 with $\mathcal{I}(Q)$.

5. Infinite points on $\mathcal{K}(Q)$ and intersection with $\mathrm{p}\mathcal{K}_{\mathrm{inf}}(Q)$

Proposition 9. *$\mathcal{K}(Q)$ meets the line at infinity at the same points as the pivotal isogonal cubic $\mathrm{p}\mathcal{K}_{\mathrm{inf}}(Q)$ with pivot **g**Q.*

Proof. This follows by writing the equation of $\mathcal{K}(Q)$ in the form

$$\sum_{\text{cyclic}} a^2\, qr\, x\, (c^2y^2 - b^2z^2) + (x+y+z) \sum_{\text{cyclic}} a^2\, p\, (c^2q - b^2r)\, yz = 0. \qquad (4)$$

Any infinite point on $\mathcal{K}(Q)$ is also a point on the cubic

$$\sum_{\text{cyclic}} a^2\, qr\, x\, (c^2y^2 - b^2z^2) = 0 \iff \sum_{\text{cyclic}} \frac{x}{p}\left(\frac{y^2}{b^2} - \frac{z^2}{c^2}\right) = 0, \qquad (5)$$

which is the pivotal isogonal cubic $\mathrm{p}\mathcal{K}_{\mathrm{inf}}(Q)$ with pivot **g**Q. □

The six other common points of $\mathcal{K}(Q)$ and $\mathrm{p}\mathcal{K}_{\mathrm{inf}}(Q)$ lie on the circumhyperbola through Q and K. They are A, B, C, Q and the two points E_1, E_2. Figure 6 shows these cubics when $Q = X_{55}$ thus **g**Q is the Gergonne point X_7. Recall that the points E_1, E_2 are X_9, X_{57}.

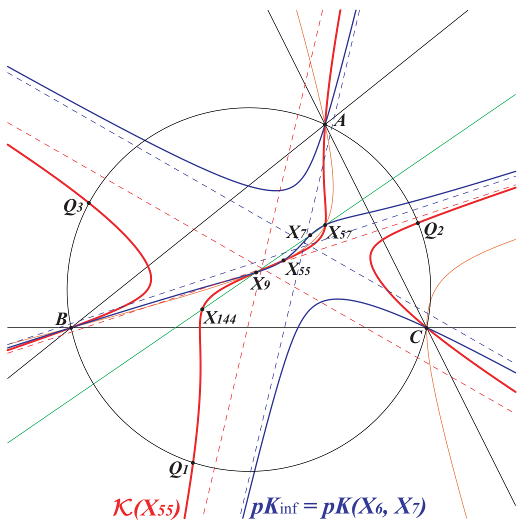

Figure 6. $\mathcal{K}(Q)$ and $\mathrm{p}\mathcal{K}_{\mathrm{inf}}(Q)$ when $Q = X_{55}$

6. $\mathcal{K}(Q)$ and the inconic with center cg$Q$

Proposition 10. *The cubic $\mathcal{K}(Q)$ contains the four foci of the inconic with center* **cg**Q *and perspector* **tg**Q.

Proof. This follows by writing the equation of $\mathcal{K}(Q)$ in the form

$$\sum_{\text{cyclic}} px(c^2q-b^2r)(c^2y^2+b^2z^2)-2\left(\sum_{\text{cyclic}} a^2(b^2-c^2)qr\right)xyz - \sum_{\text{cyclic}} px(c^2q+b^2r)(c^2y^2-b^2z^2)=0. \tag{6}$$

Indeed,

$$\sum_{\text{cyclic}} px(c^2q-b^2r)(c^2y^2+b^2z^2)-2\left(\sum_{\text{cyclic}} a^2(b^2-c^2)qr\right)xyz=0 \tag{7}$$

is the equation of the non-pivotal isogonal circular cubic $n\mathcal{K}_6(Q)$ which is the locus of foci of inconics with center on the line through G, **cg**Q and

$$\sum_{\text{cyclic}} px(c^2q+b^2r)(c^2y^2-b^2z^2)=0 \tag{8}$$

is the equation of the pivotal isogonal cubic $\mathrm{p}\mathcal{K}_6(Q)$ with pivot **cg**Q. The two cubics $\mathcal{K}(Q)$ and $\mathrm{p}\mathcal{K}_6(Q)$ obviously contain the above mentioned foci. □

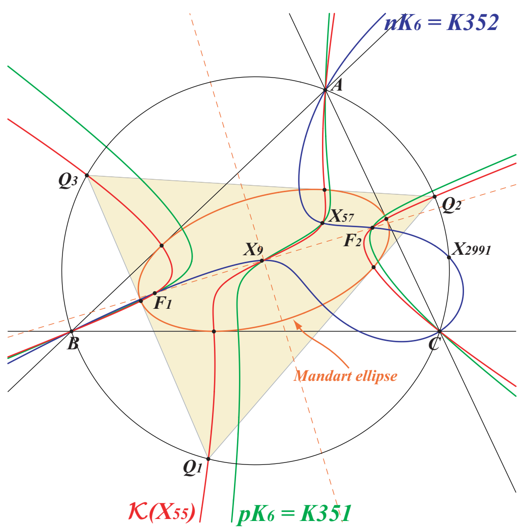

Figure 7. $\mathcal{K}(Q)$ and the related cubics $n\mathcal{K}_6(Q)$, $\mathrm{p}\mathcal{K}_6(Q)$ when $Q = X_{55}$

These two cubics generate a pencil of cubics containing $\mathcal{K}(Q)$. Note that $\mathrm{p}\mathcal{K}_6(Q)$ is a member of the pencil of isogonal pivotal cubics generated by $\mathrm{p}\mathcal{K}_{\text{inf}}(Q)$ and

$\mathrm{p}\mathcal{K}_{\mathrm{circ}}(Q)$. The root of $n\mathcal{K}_6(Q)$ is the infinite point of the trilinear polar of $\mathbf{tg}Q$. Figure 7 shows these cubics when $Q = X_{55}$. The inscribed conic is the Mandart ellipse.

In the example above, $\mathcal{K}(Q)$ contains the center **cg**Q of the inconic $\mathcal{I}(Q)$ but this is not generally true. We have

Theorem 11. *$\mathcal{K}(Q)$ contains the center* **cg**Q *of $\mathcal{I}(Q)$ if and only if Q lies on the cubic* **K172** $= \mathrm{p}\mathcal{K}(X_{32}, X_3)$.

Since we know that $\mathcal{K}(Q)$ contains the perspector **tg**Q of this same inconic when it is a pivotal cubic, it follows that there are only two cubics $\mathcal{K}(Q)$ passing through the foci, the center, the perspector of $\mathcal{I}(Q)$ and its contacts with the sidelines of ABC. These cubics are obtained when
(i) $Q = X_6$: $\mathcal{K}(X_6)$ is the Thomson cubic **K002** and $\mathcal{I}(Q)$ is the Steiner inscribed ellipse,
(ii) $Q = X_{25}$: $\mathcal{K}(X_{25})$ is **K233** $= \mathrm{p}\mathcal{K}(X_{25}, X_4)$.

In the latter case, **cg**$Q = X_6$, **tg**$Q = X_4$, **ag**$Q = X_{193}$, $\mathcal{I}(Q)$ is the K-ellipse, [1] the infinite points are those of **K169** = $\mathrm{p}\mathcal{K}(X_6, X_{69})$, the points on the circumcircle are those of $\mathrm{p}\mathcal{K}(X_6, X_{193})$. See Figure 8.

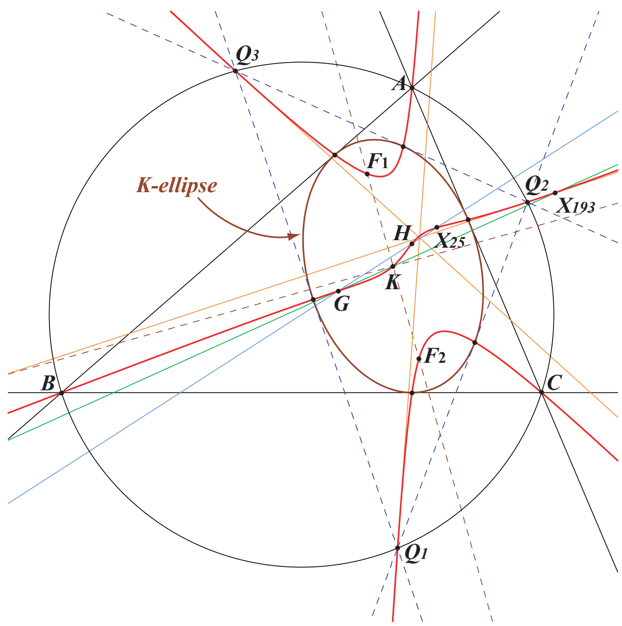

Figure 8. $\mathcal{K}(X_{25})$ and the related K-ellipse

[1]The $K-$ellipse is actually an ellipse only when triangle ABC is acute angled.

7. $\mathcal{K}(Q)$ and the Steiner ellipse

Proposition 12. *The cubic* $\mathcal{K}(Q)$ *meets the Steiner ellipse at the same points as* $\mathrm{p}\mathcal{K}(\mathbf{tg}Q,\ Q)$.

Proof. This follows by writing the equation of $\mathcal{K}(Q)$ in the form

$$\sum_{\text{cyclic}} a^2\, p\, x\, (b^2 r y^2 - c^2 q z^2) + (xy + yz + zx) \sum_{\text{cyclic}} a^2\, (b^2 - c^2)\, qr\, x = 0. \quad (9)$$

Indeed,

$$\sum_{\text{cyclic}} a^2\, p\, x\, (b^2 r y^2 - c^2 q z^2) = 0 \iff \sum_{\text{cyclic}} x \left(\frac{y^2}{c^2 q} - \frac{z^2}{b^2 r} \right) = 0 \quad (10)$$

is the equation of the pivotal cubic $\mathrm{p}\mathcal{K}(\mathbf{tg}Q,\ Q)$. $\square$

Note that $\sum_{\text{cyclic}} a^2\, (b^2 - c^2)\, qr\, x = 0$ is the equation of the line $Q\mathbf{tg}Q$. This will be construed in the next paragraph.

8. $\mathcal{K}(Q)$ and rectangular hyperbolas

Let $P = u : v : w$ be a given point and let $\mathcal{H}(P)$, $\mathcal{H}(\mathbf{g}P)$ be the two rectangular circum-hyperbolas passing through P, $\mathbf{g}P$ respectively. These have equations

$$\sum_{\text{cyclic}} u(S_B v - S_c w) yz = 0 \quad \text{and} \quad \sum_{\text{cyclic}} \left(\frac{S_B w}{c^2} - \frac{S_C v}{b^2} \right) yz = 0.$$

P must not lie on the McCay cubic in order to have two distinct hyperbolas. Indeed, $\mathbf{g}P$ lies on $\mathcal{H}(P)$ if and only if P lies on the line $O\mathbf{g}P$ i.e. P and $\mathbf{g}P$ are two isogonal conjugate points collinear with O.

Let $\mathcal{L}(Q)$ and $\mathcal{L}'(Q)$ be the two lines passing through Q with equations

$$\sum_{\text{cyclic}} a^2\, (vr(qx - py) - wq(rx - pz)) = 0$$

and

$$\sum_{\text{cyclic}} b^2 c^2 pu(v + w)(ry - qz) = 0.$$

These lines $\mathcal{L}(Q)$ and $\mathcal{L}'(Q)$ can be construed as the trilinear polars of the $Q-$isoconjugates of the infinite points of the polars of P and $\mathbf{g}P$ in the circumcircle.

The equation of $\mathcal{K}(Q)$ can be written in the form

$$\begin{aligned} &\left(\sum_{\text{cyclic}} u(S_B v - S_c w) yz \right) \left(\sum_{\text{cyclic}} a^2\, (vr(qx - py) - wq(rx - pz)) \right) \\ &= \left(\sum_{\text{cyclic}} \left(\frac{S_B w}{c^2} - \frac{S_C v}{b^2} \right) yz \right) \left(\sum_{\text{cyclic}} b^2 c^2 pu(v + w)(ry - qz) \right) \end{aligned} \quad (11)$$

which will be loosely written under the form :

$$\mathcal{H}(P) \cdot \mathcal{L}(Q) = \mathcal{H}(\mathbf{g}P) \cdot \mathcal{L}'(Q).$$

If we recall that $\mathcal{K}(Q)$ and $\mathcal{H}(P)$ have already four common points namely A, B, C, H and that $\mathcal{K}(Q)$, $\mathcal{L}(Q)$ and $\mathcal{L}'(Q)$ all contain Q, then we have

Corollary 13. *$\mathcal{K}(Q)$ meets $\mathcal{H}(P)$ again at two points on the line $\mathcal{L}'(Q)$ and $\mathcal{H}(\mathbf{g}P)$ again at two points on the line $\mathcal{L}(Q)$.*

For example, with $P = G$, $\mathcal{H}(P)$ is the Kiepert hyperbola and $\mathcal{L}'(Q)$ is the line $Q\mathbf{gt}Q$, $\mathcal{H}(\mathbf{g}P)$ is the Jerabek hyperbola and $\mathcal{L}(Q)$ is the line $Q\mathbf{tg}Q$.

9. Further representations of $\mathcal{K}(Q)$

Proposition 14. *For varying Q, the cubics $\mathcal{K}(Q)$ form a net of cubics.*

Proof. This follows by writing the equation of $\mathcal{K}(Q)$ in the form

$$\begin{aligned} &\sum_{\text{cyclic}} a^2qr\,x\left(c^2y(x+z) - b^2z(x+y)\right) = 0 \\ \iff &\sum_{\text{cyclic}} a^2qr\,x\left(x(c^2y - b^2z) - (b^2-c^2)yz\right) = 0. \end{aligned} \tag{12}$$

The equation $c^2y(x+z) - b^2z(x+y) = 0$ is that of the rectangular circumhyperbola $\mathcal{H}_A$ tangent at A to the symmedian AK. Its center is the midpoint of BC. Its sixth common point with $\mathcal{K}(Q)$ is the intersection of the lines AQ and $A_1\mathbf{ag}Q$. Thus the net is generated by the three decomposed cubics which are the union of a sideline of ABC and the corresponding hyperbola such as $\mathcal{H}_A$. □

Proposition 15. *$\mathcal{K}(Q)$ is a pivotal cubic $\mathrm{p}\mathcal{K}(Q)$ if and only if Q lies on the circumhyperbola $\mathcal{H}$ passing through G and K.*

Proof. We write the equation of $\mathcal{K}(Q)$ in the form

$$\sum_{\text{cyclic}} b^2c^2\,p\,x\,(ry^2 - qz^2) + \left(\sum_{\text{cyclic}} a^2(b^2-c^2)qr\right) xyz = 0. \tag{13}$$

Recall that $\mathcal{K}(Q)$ meets the sidelines of triangle ABC again at the vertices of the cevian triangle of $\mathbf{tg}Q$. Thus, the cubic is a pivotal cubic is and only if the term in xyz vanishes. It is now sufficient to observe that the equation of the hyperbola $\mathcal{H}$ is $\sum_{\text{cyclic}} a^2(b^2-c^2)yz = 0.$ □

See a more detailed study of these $\mathrm{p}\mathcal{K}(Q)$ in §10.1.

Proposition 16. *The cubic $\mathcal{K}(Q)$ belongs to another pencil of similar cubics generated by another pivotal cubic and another isogonal non-pivotal cubic.*

Proof.

$$\sum_{\text{cyclic}} px(c^2q - b^2r)(c^2y^2 + b^2z^2) - \left(\sum_{\text{cyclic}} a^2(b^2 - c^2)qr\right) xyz + \sum_{\text{cyclic}} a^4qr(y-z)yz = 0. \tag{14}$$

Indeed,

$$\sum_{\text{cyclic}} px(c^2q - b^2r)(c^2y^2 + b^2z^2) - \left(\sum_{\text{cyclic}} a^2(b^2 - c^2)qr\right) xyz = 0 \tag{15}$$

is the equation of the non-pivotal isogonal cubic $n\mathcal{K}_7(Q)$ with root the infinite point of the trilinear polar of $\mathbf{tg}Q$ again and

$$\sum_{\text{cyclic}} a^4qr(y-z)yz = 0 \tag{16}$$

is the equation of the pivotal cubic $\mathrm{p}\mathcal{K}_7(Q)$ with pivot the centroid G and pole the $X_{32}-$isoconjugate of Q i.e. the point **gtg**Q. □

The cubics $n\mathcal{K}_6(Q)$ and $n\mathcal{K}_7(Q)$ obviously coincide when Q lies on the circumhyperbola $\mathcal{H}$ passing through G and K. Figure 9 shows these cubics when $Q = X_{55}$.

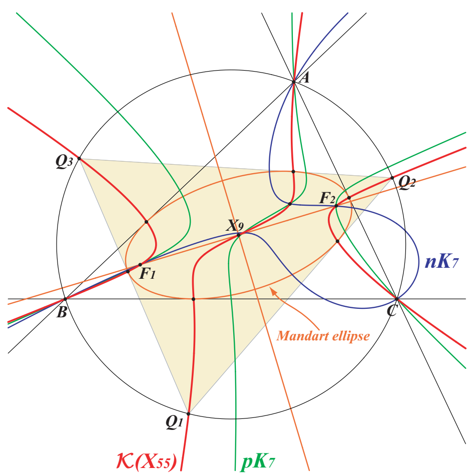

Figure 9. $\mathcal{K}(Q)$ and the related cubics $n\mathcal{K}_7(Q)$, $\mathrm{p}\mathcal{K}_7(Q)$ when $Q = X_{55}$

10. Special cubics $\mathcal{K}(Q)$

10.1. *Pivotal cubics* $\mathrm{p}\mathcal{K}(Q)$. Recall that for any point Q on the circumhyperbola $\mathcal{H}$ passing through G and K the cubic $\mathcal{K}(Q)$ becomes a pivotal cubic with pole Q and pivot $\mathbf{tg}Q$ on the Kiepert hyperbola. In this case, $\mathcal{K}(Q)$ has equation :

$$\sum_{\text{cyclic}} b^2c^2\,p\,x\,(ry^2 - qz^2) = 0 \iff \sum_{\text{cyclic}} \frac{x}{a^2}\left(\frac{y^2}{q} - \frac{z^2}{r}\right) = 0 \qquad (17)$$

The isopivot (secondary pivot) is clearly the Lemoine point K since the tangents at A, B, C are the symmedians. The points $\mathbf{g}Q$ and $\mathbf{ag}Q$ lie on the line GK namely the tangent at G to the Kiepert hyperbola.

These cubics form a pencil of pivotal cubics passing through A, B, C, G, H, K and tangent to the symmedians. Recall that they have the remarkable property to intersect the circumcircle at three other points Q_1, Q_2, Q_3 with concurrent tangents such that $\mathbf{ag}Q$ is the orthocenter of $Q_1Q_2Q_3$. See [4] for further informations.

This pencil is generated by the Thomson cubic **K002** (the only isogonal cubic) and by **K141** (the only isotomic cubic). See **CL043** in [2] for a selection of other cubics of the pencil among them **K273**, the only circular cubic, and **K233** seen above.

10.2. *Circular cubics* $\mathcal{K}(Q)$. We have seen that $\mathcal{K}(Q)$ meets the line at infinity at the same points as the pivotal isogonal cubic $\mathrm{p}\mathcal{K}_{\mathrm{inf}}(Q)$ with pivot $\mathbf{g}Q$. It easily follows that $\mathcal{K}(Q)$ is a circular cubic if and only if $\mathrm{p}\mathcal{K}_{\mathrm{inf}}(Q)$ is itself a circular cubic therefore if and only if $\mathbf{g}Q$ lies at infinity hence Q must lie on the circumcircle $\mathcal{C}$. Thus, we have :

Theorem 17. *For any point Q on the circumcircle, $\mathcal{K}(Q)$ is a circular cubic with singular focus on the circle with center O and radius $2R$. The tangent at Q always passes through O.*

The real asymptote envelopes a deltoid, the homothetic of the Steiner deltoid under $h(G, 4)$. See Figure 10.

For example, **K273** (obtained for $Q = X_{111}$, the Parry point) and **K306** (obtained for $Q = X_{759}$) are two cubics of this type in [2]. See also the bottom of the page **CL035** in [2].

10.3. *Lemoine generalized cubics* $\mathcal{K}(Q)$. A necessary (but not sufficient) condition to obtain a Lemoine generalized cubic $\mathcal{K}(Q)$ is that the cevian triangle of $\mathbf{tg}Q$ must be a pedal triangle. Hence, $\mathbf{tg}Q$ must be a point on the Lucas cubic **K007** therefore Q must be on its isogonal transform **K172**.

The only identified points that give a Lemoine generalized cubic are H and X_{56}.

$\mathcal{K}(H)$ is **K028**, the third Musselman cubic. It is also the only cubic with asymptotes making 60° angles with one another i.e. the only equilateral cubic of this type.

$\mathcal{K}(X_{56})$ is **K360**, at the origin of this note. See Figure 11.

The conic inscribed in the triangles ABC and $Q_1Q_2Q_3$ is the incircle of ABC since $\mathbf{tg}X_{56}$ is the Gergonne point X_7. $Q_1Q_2Q_3$ is a poristic triangle.

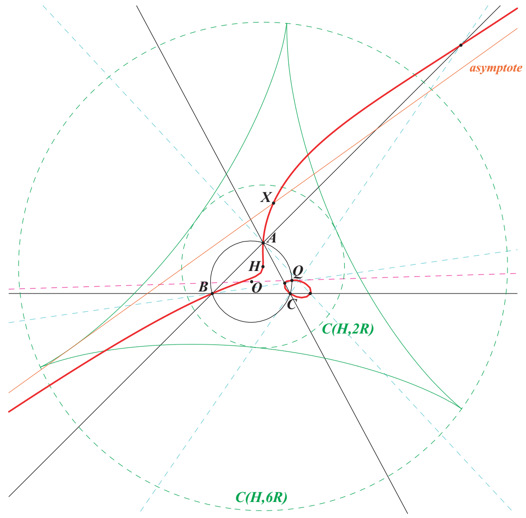

Figure 10. Circular cubics $\mathcal{K}(Q)$ and deltoid

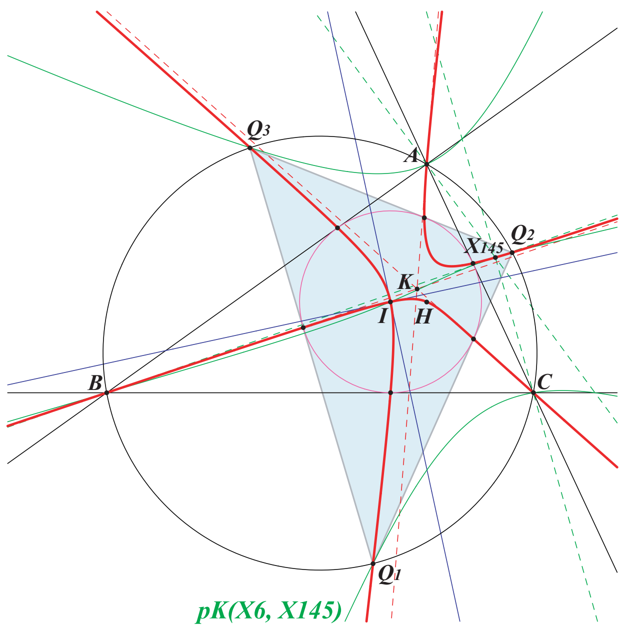

Figure 11. The Lemoine generalized cubic $\mathcal{K}(X_{56})$ = **K360**

10.4. $\mathcal{K}(X_{32})$. $\mathcal{K}(X_{32})$ has the remarkable property to have its six tangents at its common points with the circumcircle concurrent at the Lemoine point K. It follows that the triangles ABC and $Q_1Q_2Q_3$ have the same Lemoine point and the same Brocard axis. The polar conic of K is therefore the circumcircle.

The satellite conic of the circumcircle is the Brocard ellipse whose real foci Ω_1, Ω_2 (Brocard points) lie on the cubic. See Figure 12.

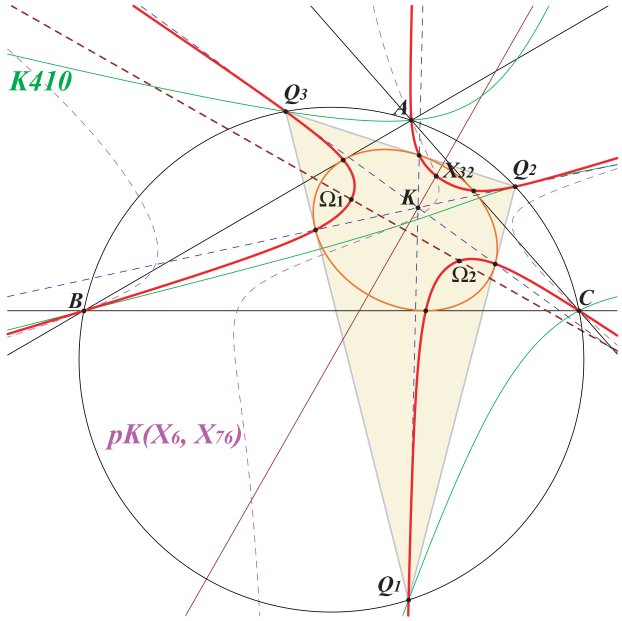

Figure 12. The cubic $\mathcal{K}(X_{32})$

Remark. $\mathcal{K}(X_{32})$ belongs to a pencil of circum-cubics having the same property to meet the circumcircle at six points A, B, C, Q_1, Q_2, Q_3 with tangents concurring at K hence the polar conic of K is always the circumcircle.

The cubic of the pencil passing through the given point $P = u : v : w$ has an equation of the form

$$\sum_{\text{cyclic}} a^4 vwyz \left((c^2v - b^2w)x - (u(c^2y - b^2z)\right) = 0,$$

which shows that the pencil is generated by three decomposed cubics, one of them being the union of the sidelines AB, AC and the line joining P to the feet K_a of the $A-$symmedian, the other two similarly. Each cubic meets the Brocard ellipse at six points which are the tangentials of the six points above. Three of them are K_a, K_b, K_c and the other points are the contacts of the Brocard ellipse with the sidelines of $Q_1Q_2Q_3$.

10.5. $\mathcal{K}(X_{54})$. $\mathcal{K}(X_{54}) =$ **K361** is the only cubic of the family meeting the circumcircle at the vertices of an equilateral triangle $Q_1Q_2Q_3$ namely the circumnormal triangle. The tangents at these points concur at O. **K361** is the isogonal transform of **K026**, the (first) Musselman cubic and the locus of pivots of pivotal cubics that pass through the vertices of the circumnormal triangle. See Figure 13 and further details in [2].

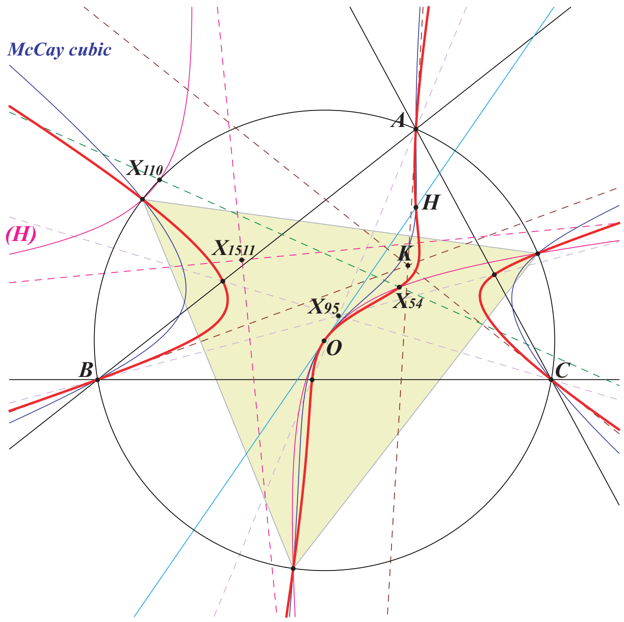

Figure 13. The cubic $\mathcal{K}(X_{54}) =$ **K361**

10.6. *$\mathcal{K}(Q)$ with concurring asymptotes.* $\mathcal{K}(Q)$ has three (not necessarily all real) concurring asymptotes if and only if Q lies on a circumcubic passing through O, H, X_{140}. This latter cubic is a $\mathcal{K}_{60}^{+}$ i.e. it has three real concurring asymptotes making 60° angles with one another. These are the parallels at X_{547} (the midpoint of X_2, X_5) to those of the McCay cubic **K003**. The cubic meets the cirumcircle at the same points as $\mathrm{p}\mathcal{K}(X_6, X_{140})$ where X_{140} is the midpoint of X_3, X_5. See Figure 14.

The two cubics $\mathcal{K}(H) =$ **K028** and $\mathcal{K}(X_{140})$ have concurring asymptotes but their common point is not on the curve. These are $\mathcal{K}^{+}$ cubics.

On the contrary, $\mathcal{K}(X_3)$ is a central cubic and the asymptotes meet at O on the curve. It is said to be a $\mathcal{K}^{++}$ cubic. See Figure 15.

11. Isogonal transform of $\mathcal{K}(Q)$

Under isogonal conjugation with respect to ABC, $\mathcal{K}(Q)$ is transformed into another circum-cubic **g**$\mathcal{K}(Q)$ meeting $\mathcal{K}(Q)$ again at the four foci of $\mathcal{I}(Q)$ and at the two points E_1, E_2 intersections of the line $G\mathbf{ag}Q$ with the conic $ABCKQ$.

Thus, $\mathcal{K}(Q)$ and **g**$\mathcal{K}(Q)$ have nine known common points. When they are distinct i.e. when Q is not K i.e. when $\mathcal{K}(Q)$ is not the Thomson cubic, they generate a pencil of cubics which contains $\mathrm{p}\mathcal{K}(X_6, \mathbf{cg}Q)$.

It is easy to verify that **g**$\mathcal{K}(Q)$

(i) contains the circumcenter O, **g**Q, the midpoints of ABC,

(ii) is tangent at A, B, C to the cevian lines of the $X_{32}-$isoconjugate of Q i.e. the point **gtg**Q,

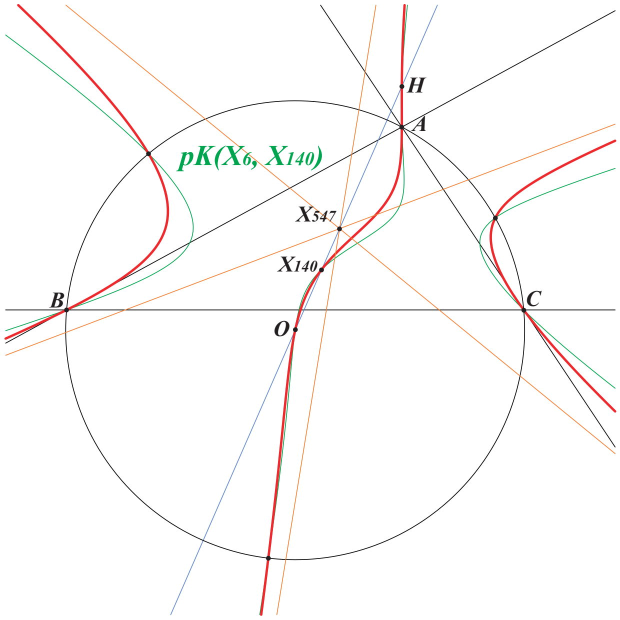

Figure 14. The cubic $\mathcal{K}^+_{60}$

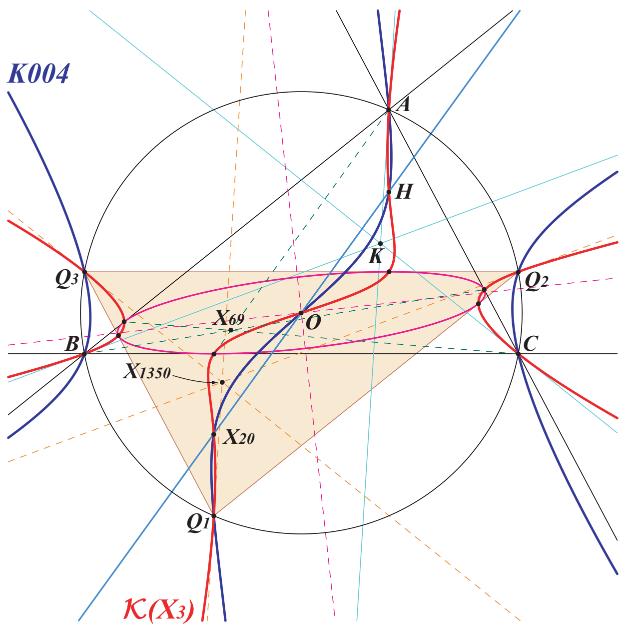

Figure 15. The cubic $\mathcal{K}(X_3)$

(iii) meets the circumcircle at the same points as $\mathrm{p}\mathcal{K}(X_6, \mathbf{g}Q)$ hence the orthocenter of the triangle $O_1O_2O_3$ formed by these points is $\mathbf{g}Q$; following a result of [4],

the inconic with perspector **tcg**Q is inscribed in ABC and $O_1O_2O_3$,
(iv) has the same asymptotic directions as $\mathrm{p}\mathcal{K}(X_6, \mathbf{ag}Q)$.

Except the case $Q = K$, **g**$\mathcal{K}(Q)$ cannot be a cubic of type $\mathcal{K}(Q)$.

The tangents to **g**$\mathcal{K}(Q)$ at A, B, C are still concurrent (at **gtg**Q) but in general, the tangents at the other intersections of **g**$\mathcal{K}(Q)$ with the circumcircle are not now concurrent unless Q lies on a circular circum-quartic which is the isogonal transform of **Q063**. This quartic contains X_1, X_3, X_6, X_{64}, X_{2574}, X_{2575}, the excenters.

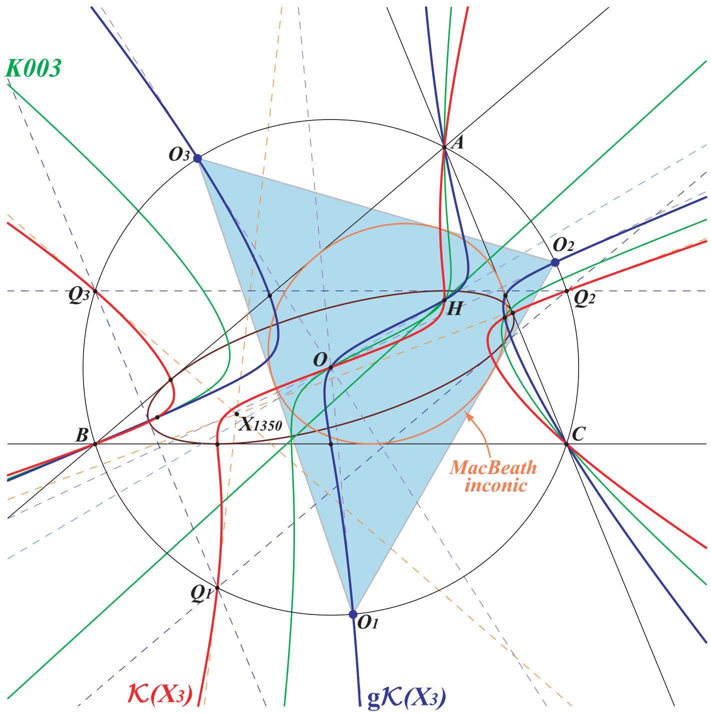

Figure 16. $\mathcal{K}(X_3)$, **g**$\mathcal{K}(X_3)$ and **K003**

Figure 16 presents $\mathcal{K}(X_3)$ and **g**$\mathcal{K}(X_3)$. These two cubics generate a pencil which contains the McCay cubic **K003** and the Euler isogonal focal cubic **K187**. The nine common points of these four cubics are A, B, C, O, H and the four foci of the inscribed conic with center O.

g$\mathcal{K}(X_3)$ meets the circumcircle at the same points O_1, O_2, O_3 as the Orthocubic **K006** and the triangles ABC, $O_1O_2O_3$ share the same orthocenter H therefore the same Euler line. The tangents at O_1, O_2, O_3 concur at O and those at A, B, C concur at X_{25}. The MacBeath inconic (with center X_5, foci O and H) is inscribed in ABC and $O_1O_2O_3$.

g$\mathcal{K}(X_3)$ meets the line at infinity at the same points as the Darboux cubic **K004**. Hence, its three asymptotes are parallel to the altitudes of ABC.

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