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Isocubics with Concurrent Normals

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Abstract. It is well known that the tangents at A, B, C to a pivotal isocubic concur. This paper studies the situation where the normals at the same points concur. The case of non-pivotal isocubics is also considered.

1. Pivotal isocubics

Consider a pivotal isocubic $\mathrm{p}\mathcal{K} = \mathrm{p}\mathcal{K}(\Omega, P)$ with pole $\Omega = p : q : r$ and pivot P, *i.e.*, the locus of point M such as P, M and its $\Omega-$isoconjugate M^* are collinear. This has equation

$$ux(ry^2 - qz^2) + vy(pz^2 - rx^2) + wz(qx^2 - py^2) = 0.$$

It is well known that the tangents at A, B, C and P to $\mathrm{p}\mathcal{K}$, being respectively the lines $-\frac{v}{q}y + \frac{w}{r}z = 0$, $\frac{u}{p}x - \frac{w}{r}z = 0$, $-\frac{u}{p}x + \frac{v}{q}y = 0$, concur at $P^* = \frac{p}{u} : \frac{q}{v} : \frac{r}{w}$. [1] We characterize the pivotal cubics whose normals at the vertices A, B, C concur at a point. These normals are the lines

$$\begin{aligned}
\mathsf{n}A: &\quad (S_A rv + (S_A + S_B)qw)y + (S_A qw + (S_C + S_A)rv)z = 0,\\
\mathsf{n}B: &\quad (S_B ru + (S_A + S_B)pw)x + (S_B pw + (S_B + S_C)ru)z = 0,\\
\mathsf{n}C: &\quad (S_C qu + (S_C + S_A)pv)x + (S_C pv + (S_B + S_C)qu)y = 0.
\end{aligned}$$

These three normals are concurrent if and only if

$$(pvw + qwu + ruv)(a^2qru + b^2rpv + c^2pqw) = 0.$$

Let us denote by $\mathcal{C}_\Omega$ the circumconic with perspector Ω, and by $\mathcal{L}_\Omega$ the line which is the $\Omega-$isoconjugate of the circumcircle. [2] These have barycentric equations

$$C_\Omega: \quad pyz + qzx + rxy = 0,$$

and

$$\mathcal{L}_\Omega: \quad \frac{a^2}{p}x + \frac{b^2}{q}y + \frac{c^2}{r}z = 0.$$

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[1] The tangent at P, namely, $u(rv^2 - qw^2)x + v(pw^2 - ru^2)y + w(qu^2 - pv^2)z = 0$, also passes through the same point.

[2] This line is also the trilinear polar of the isotomic conjugate of the isogonal conjugate of Ω.

Theorem 1. *The pivotal cubic* $\mathrm{p}\mathcal{K}(\Omega, P)$ *has normals at* A, B, C *concurrent if and only if*
(1) P *lies on* $\mathcal{C}_\Omega$*, equivalently,* P^* *lies on the line at infinity, or*
(2) P *lies on* $\mathcal{L}_\Omega$*, equivalently,* P^* *lies on the circumcircle.*

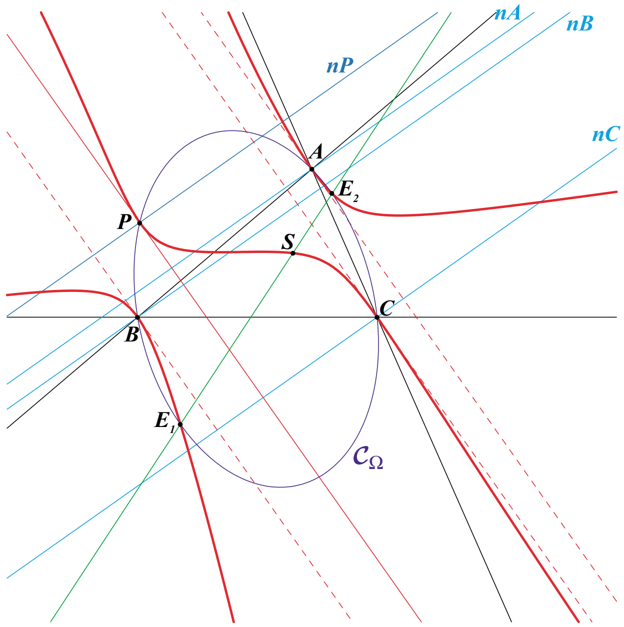

Figure 1. Theorem 1(1): $\mathrm{p}\mathcal{K}$ with concurring normals

More precisely, in (1), the tangents at A, B, C are parallel since P^* lies on the line at infinity. Hence the normals are also parallel and "concur" at X on the line at infinity. The cubic $\mathrm{p}\mathcal{K}$ meets $\mathcal{C}_\Omega$ at A, B, C, P and two other points E_1, E_2 lying on the polar line of P^* in $\mathcal{C}_\Omega$, *i.e.*, the conjugate diameter of the line PP^* in $\mathcal{C}_\Omega$. Obviously, the normal at P is parallel to these three normals. See Figure 1.

In (2), P^* lies on the circumcircle and the normals concur at X, antipode of P^* on the circumcircle. $\mathrm{p}\mathcal{K}$ passes through the (not always real) common points E_1, E_2 of $\mathcal{L}_\Omega$ and the circumcircle. These two points are isoconjugates. See Figure 2.

2. The orthopolar

The tangent $\mathtt{t}M$ at any non-singular point M to any curve is the polar line (or first polar) of M with respect to the curve and naturally the normal $\mathtt{n}M$ at M is the perpendicular at M to $\mathtt{t}M$. For any point M not necessarily on the curve, we define the *orthopolar* of M with respect to the curve as the perpendicular at M to the polar line of M.

In Theorem 1(1) above, we may ask whether there are other points on $\mathrm{p}\mathcal{K}$ such that the normal passes through X. We find that the locus of point Q such that the orthopolar of Q contains X is the union of the line at infinity and the circumconic

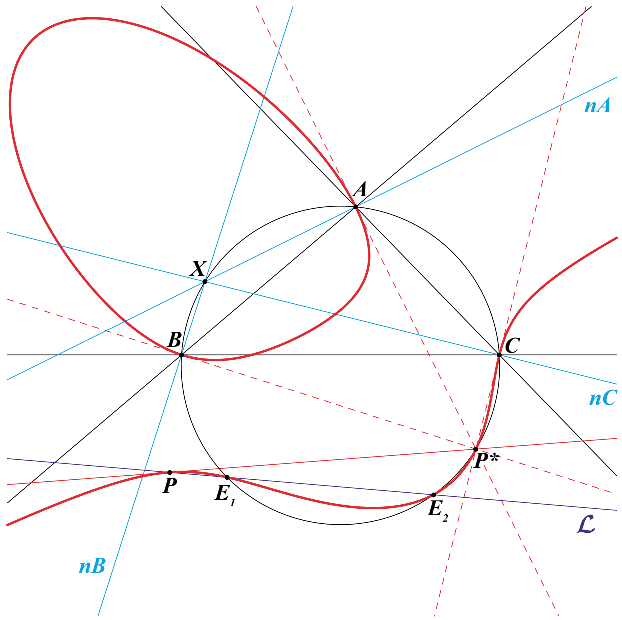

Figure 2. Theorem 1(2): p$\mathcal{K}$ with concurring normals

passing through P and P^*, the isoconjugate of the line PP^*. Hence, there are no other points on the cubic with normals passing through X.

In Theorem 1(2), the locus of point Q such that the orthopolar of Q contains X is now a circum-cubic (K) passing through P^* and therefore having six other (not necessarily real) common points with p$\mathcal{K}$. Figure 3 shows p$\mathcal{K}(X_2, X_{523})$ where four real normals are drawn from the Tarry point X_{98} to the curve.

3. Non-pivotal isocubics

Lemma 2. *Let M be a point and m its trilinear polar meeting the sidelines of ABC at U, V, W. The perpendiculars at A, B, C to the lines AU, BV, CW concur if and only if M lies on the Thomson cubic. The locus of the point of concurrence is the Darboux cubic.*

Let us now consider a non-pivotal isocubic n$\mathcal{K}$ with pole $\Omega = p : q : r$ and root[3] $P = u : v : w$. This cubic has equation :

$$ux(ry^2 + qz^2) + vy(pz^2 + rx^2) + wz(qx^2 + py^2) + kxyz = 0.$$

Denote by n$\mathcal{K}_0$ the corresponding cubic without xyz term, *i.e.*,

$$ux(ry^2 + qz^2) + vy(pz^2 + rx^2) + wz(qx^2 + py^2) = 0.$$

[3]An n$\mathcal{K}$ meets again the sidelines of triangle ABC at three collinear points U, V, W lying on the trilinear polar of the root.

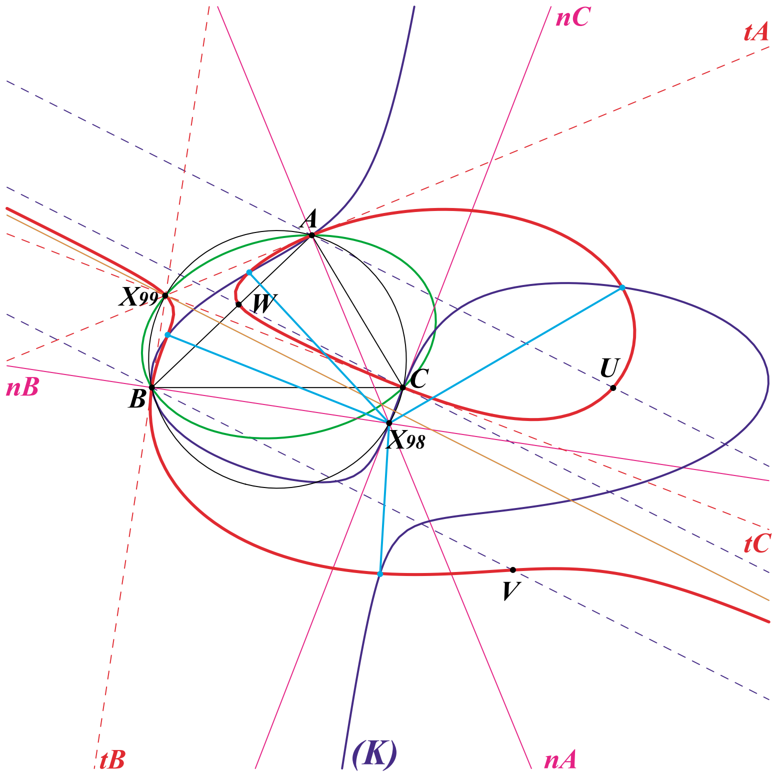

Figure 3. Theorem 1(2): Other normals to p$\mathcal{K}$

It can easily be seen that the tangents tA, tB, tC do not depend of k and pass through the feet U', V', W' of the trilinear polar of P^* [4]. Hence it is enough to take the cubic n$\mathcal{K}_0$ to study the normals at A, B, C.

Theorem 3. *The normals of* n$\mathcal{K}_0$ *at* A, B, C *are concurrent if and only if*
(1) Ω *lies on the pivotal isocubic* p$\mathcal{K}_1$ *with pole* $\Omega_1 = a^2u^2 : b^2v^2 : c^2w^2$ *and pivot* P*, or*
(2) P *lies on the pivotal isocubic* p$\mathcal{K}_2$ *with pole* $\Omega_2 = \frac{p^2}{a^2} : \frac{q^2}{b^2} : \frac{r^2}{c^2}$ *and pivot* $P_2 = \frac{p}{a^2} : \frac{q}{b^2} : \frac{r}{c^2}$.

p$\mathcal{K}_1$ is the p$\mathcal{K}$ with pivot the root P of the n$\mathcal{K}_0$ which is invariant in the isoconjugation which swaps P and the isogonal conjugate of the isotomic conjugate of P.

By Lemma 2, it is clear that p$\mathcal{K}_2$ is the $\Omega-$isoconjugate of the Thomson cubic.

The following table gives a selection of such cubics p$\mathcal{K}_2$. Each line of the table gives a selection of n$\mathcal{K}_0(\Omega, X_i)$ with concurring normals at A, B, C.

[4]In other words, these tangents form a triangle perspective to ABC whose perspector is P^*. Its vertices are the harmonic associates of P^*.

Cubic	Ω	Ω_2	P_2	X_i on the curve for $i =$
$K034$	X_1	X_2	X_{75}	1, 2, 7, 8, 63, 75, 92, 280, 347, 1895
$K184$	X_2	X_{76}	X_{76}	2, 69, 75, 76, 85, 264, 312
$K099$	X_3	X_{394}	X_{69}	2, 3, 20, 63, 69, 77, 78, 271, 394
	X_4	X_{2052}	X_{264}	2, 4, 92, 253, 264, 273, 318, 342
	X_9	X_{346}	X_{312}	2, 8, 9, 78, 312, 318, 329, 346
	X_{25}	X_{2207}	X_4	4, 6, 19, 25, 33, 34, 64, 208, 393
$K175$	X_{31}	X_{32}	X_1	1, 6, 19, 31, 48, 55, 56, 204, 221, 2192
$K346$	X_{32}	X_{1501}	X_6	6, 25, 31, 32, 41, 184, 604, 2199
	X_{55}	X_{220}	X_8	1, 8, 9, 40, 55, 200, 219, 281
	X_{56}	X_{1407}	X_7	1, 7, 56, 57, 84, 222, 269, 278
	X_{57}	X_{279}	X_{85}	2, 7, 57, 77, 85, 189, 273, 279
	X_{58}	X_{593}	X_{86}	21, 27, 58, 81, 86, 285, 1014, 1790
	X_{75}	X_{1502}	X_{561}	75, 76, 304, 561, 1969

For example, all the isogonal $\mathsf{n}\mathcal{K}_0$ with concurring normals must have their root on the Thomson cubic. Similarly, all the isotomic $\mathsf{n}\mathcal{K}_0$ with concurring normals must have their root on $K184 = \mathrm{p}\mathcal{K}(X_{76}, X_{76})$.

Figure 4 shows $\mathsf{n}\mathcal{K}_0(X_1, X_{75})$ with normals concurring at O. It is possible to draw from O six other (not necessarily all real) normals to the curve. The feet of these normals lie on another circum-cubic labeled (K) in the figure.

In the special case where the non-pivotal cubic is a singular cubic $\mathrm{c}\mathcal{K}$ with singularity F and root P, the normals at A, B, C concur at F if and only if F lies on the Darboux cubic. Furthermore, the locus of M whose orthopolar passes through F being also a nodal circumcubic with node F, there are two other points on $\mathrm{c}\mathcal{K}$ with normals passing through F. In Figure 5, $\mathrm{c}\mathcal{K}$ has singularity at O and its root is X_{394}. The corresponding nodal cubic passes through the points O, X_{25}, X_{1073}, X_{1384}, X_{1617}. The two other normals are labelled OR and OS.

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[4] C. Kimberling, *Encyclopedia of Triangle Centers*, available at http://faculty.evansville.edu/ck6/encyclopedia/ETC.html.

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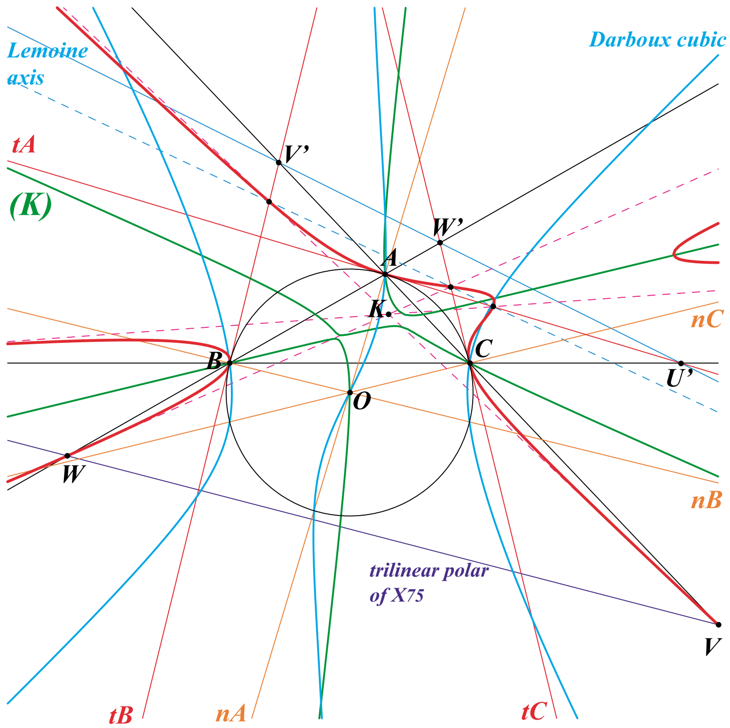

Figure 4. $\mathsf{n}\mathcal{K}_0(X_1, X_{75})$ with normals concurring at O

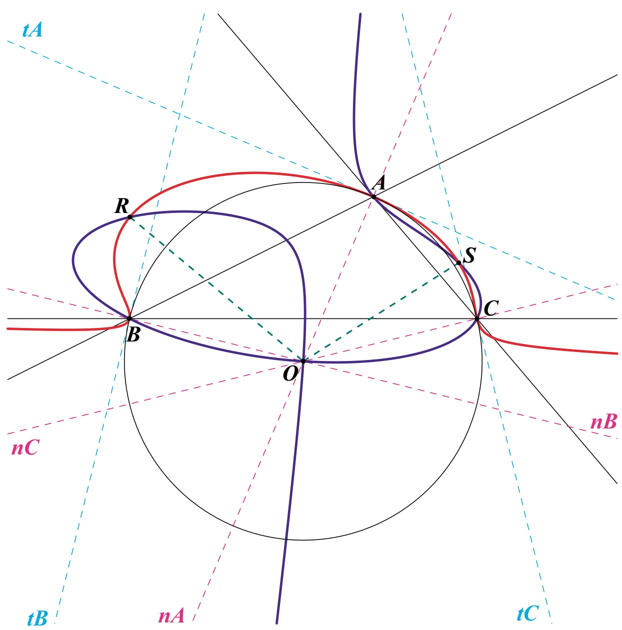

Figure 5. A $c\mathcal{K}$ with normals concurring at O