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The Cevian Simson Transformation

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Abstract. We study a transformation whose origin lies in the relation between concurrent Simson lines parallel to cevian lines as seen in [4].

1. Introduction

Let $M = (u : v : w)$ be a point. In [4] we raised the following question: to find a point P such that the three Simson lines passing through P are parallel to the three cevian lines of M.

The answer to this question is that M must lie on the McCay cubic **K003** and, in this case, the corresponding point P is given by

$$P = \left(\frac{u^2(v+w)}{a^2} : \frac{v^2(w+u)}{b^2} : \frac{w^2(u+v)}{c^2} \right),$$

In this case one can find an isogonal pivotal cubic whose asymptotes are also parallel to the cevian lines of M.

We note the strong connection with the cubic **K024** whose equation is

$$\sum_{\text{cyclic}} \frac{x^2(y+z)}{a^2} = 0.$$

When M lies on **K024**, P lies on the line at infinity.

If we denote by gM, tM, cM, aM the isogonal conjugate, the isotomic conjugate, the complement, the anticomplement of M respectively then $P = tgM \times ctM$ where $\times$ is the barycentric product. $\mathcal{L}(M)$ will denote the trilinear polar of M.

In this paper, we extend to the whole plane the mapping CST that sends M onto P which we call the Cevian Simson Transformation.

2. Properties of CST

2.1. *Singular points and consequences.*

Proposition 1. CST *has six singular points which are A, B, C each counted twice.*

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This is obvious from the coordinates of P. It follows that CST transforms any curve $\mathcal{C}$ of degree n into a curve $\mathcal{C}'$ of degree $3n$ which must be reduced according to the number and the nature of the singular points on the original curve.

More precisely, let $G_aG_bG_c$ be the antimedial triangle.

(1) If $\mathcal{C}$ contains only A and is not tangent to G_bG_c, the degree of $\mathcal{C}'$ is $3n-1$,
(2) If $\mathcal{C}$ contains A, B, C and is not tangent at these points to a sideline of $G_aG_bG_c$, the degree of $\mathcal{C}'$ is $3n-3$.
(3) If $\mathcal{C}$ contains A, B, C and has a double contact at these points to a sideline of $G_aG_bG_c$, the degree of $\mathcal{C}'$ is $3n-6$.

In particular,

(4) The transform of a line is generally a cubic which must be tangent to the sidelines of ABC. See §3 below.
(5) The transform of a circum-conic is generally a circum-cubic. See §4 below.
(6) The transform of a circum-cubic tangent at A, B, C to the sidelines of $G_aG_bG_c$ is generally a circum-cubic.

A very special case: the Steiner ellipse is tangent A, B, C to the sidelines of $G_aG_bG_c$ hence its transform is a "curve" of degree 0, namely a point. This point is actually X_{76}, the isotomic conjugate of the Lemoine point $K = X_6$. Note that X_{76} is also CST(X_2).

Consequently, the curve $\mathcal{C}'$ above will have a singular point at X_{76} whose multiplicity is $2n$ lowered according to the singular points on $\mathcal{C}$ as above. The nature of this singular point, i.e., the reality of the nodal tangents, will depend of the nature of the intersections of $\mathcal{C}$ and the Steiner ellipse. If $\mathcal{C}$ contains X_2, the multiplicity must be increased.

This will be developed in the following sections.

2.2. *Fixed points.*

Proposition 2. CST *has one and only one fixed point which is the orthocenter* $H = X_4$ *of* ABC.

Indeed, M is a fixed point of CST if and only if $P = M \iff ctM = X_6 \iff M = X_4$. It follows that the transform $\mathcal{C}'$ of any curve $\mathcal{C}$ passing through H also passes through H.

2.3. *Some special* CST *images.*

G_a, G_b, G_c are transformed into A, B, C.

The infinite points of the sidelines of ABC are transformed into the traces of the de Longchamps axis $\mathcal{L}(X_{76})$ on these same sidelines.

The infinite points of **K003** are transformed into the cusps of the Steiner deltoid $\mathcal{H}_3$.

The infinite points of an equilateral cubic whose asymptotes are not parallel to those of **K024** are transformed into the cusps of a deltoid inscribed in ABC.

If these asymptotes are parallel to those of **K024**, their infinite points are transformed into the infinite points of the sidelines of ABC.

2.4. *Pre-images of a point.* We already know that X_{76} has infinitely many pre-images which are $G = X_2$ and the points on the Steiner ellipse and also $H = X_4$, being a fixed point, has already at least one pre-image namely itself.

We consider a point P different of X_{76} and not lying on a sideline of ABC or $G_aG_bG_c$. We wish to characterize all the points M such that CST(M) is P.

When expressing that CST$(M) = P$ we obtain three equations representing three nodal circum-cubic curves with nodes at A, B, C. Their isogonal transforms are three conics each passing through one vertex of ABC. These conics have generally three common points hence P has three pre-images M_1, M_2, M_3.

The nature of these points (real or not, distinct or not) depends of the position of P with respect to the sidelines of ABC, the cevian lines of X_{76} and mainly the Ehrmann-MacBeath cubic **K244** which is the locus of the cusps of all the deltoids inscribed in ABC and also the CST image of the line at infinity. For more informations about **K244**, see [1].

More precisely, see Figure 1,

(i) when P lies inside the yellow region (excluding its "edges" mentioned above) there are three real distinct points M_1, M_2, M_3;

(ii) when M lies outside, there is only one real point;

(iii) when P lies on **K244** (but not on the other lines above), there is only one point (counted twice) and this point lies on the line at infinity. For example, when $P = X_{764}$ we obtain X_{513}. This will be detailed in section 3.

The net generated by the three conics above contains the circum-conic which is the isogonal transform of the line passing through X_2 and gtP always defined since $gtP \neq X_2$. This line must contain the points M_1, M_2, M_3.

On the other hand, each cubic which is the union of one conic and the opposite sideline of ABC must contain the isogonal conjugates of the points M_1, M_2, M_3. Hence the three isogonal transforms of these three cubics contain M_1, M_2, M_3. These three latter cubics generate a pencil which contains several simple cubics and, in particular, the $n\mathcal{K}_0(\Omega, \Omega)$ where Ω is the isogonal conjugate of the infinite point of the trilinear polar of tP, a point clearly on the circumcircle of ABC.

This cubic is a member of the class **CL026** and has always three concurring asymptotes and is tritangent at A, B, C to the Steiner ellipse unless it decomposes. See Example 3 below.

The equation of $n\mathcal{K}_0(\Omega, \Omega)$

$$\sum_{\text{cyclic}} (v - w)\, \frac{x^2(y + z)}{a^2} = 0$$

clearly shows that its CST image is the line X_2, P. See §5 for more details.

Furthermore, if the coordinates of $X_2 + \lambda\, gtP$ are inserted into the equation of $n\mathcal{K}_0(\Omega, \Omega)$ then the 3rd degree polynomial in λ has no term in λ^2. Hence the sum of the three values corresponding to the points M_1, M_2, M_3 is zero. It follows that the isobarycenter of M_1, M_2, M_3 is X_2.

In conclusion and generally speaking, we have

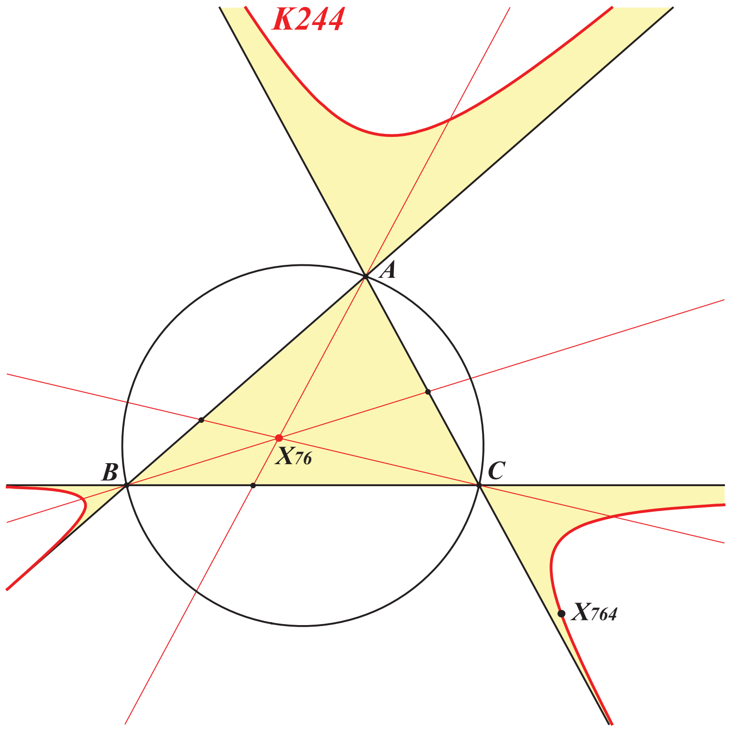

Figure 1. Regions delimited by the cubic **K244**

Proposition 3. (a) *The pre-images* M_1, M_2, M_3 *of a point* $P \neq X_{76}$ *are the intersections of the line joining* X_2 *and* gtP *with the cubic* $n\mathcal{K}_0(\Omega, \Omega)$.
(b) *The* CST *image of this line is the line* X_2, P.
(c) *The centroid* G *of* ABC *is the isobarycenter of* M_1, M_2, M_3.

Example 1. *With* $P = X_4$*, we find the Euler line and* $n\mathcal{K}_0(X_{112}, X_{112})$*. Hence the pre-images of* X_4 *are* X_4, X_{1113}, X_{1114}.

Example 2. *With* $P = X_3$*, we find the line through* X_2, X_{98}, X_{110}*, etc, and* $n\mathcal{K}_0(X_{112}, X_{112})$ *again. One of the pre-images is* X_{110} *and the other are real when* ABC *is acute angled.*

Example 3. *The cubic* $n\mathcal{K}_0(\Omega, \Omega)$ *contains* X_2 *if and only if* P *lies on the line* X_2X_{76}*. In this case, it splits into the Steiner ellipse and the line* X_2X_6.

2.5. CST *images of cevian triangles.* Let $P_aP_bP_c$ be the cevian triangle of $P = (p : q : r)$ and let $Q_aQ_bQ_c$ be its anticomplement.

We have $P_a = (0 : q : r)$, $Q_a = (q + r : -q + r : q - r)$. It is easy to see that $\text{CST}(P_a) = \text{CST}(Q_a) = (0 : c^2q : b^2r) = R_a$. The points R_b, R_c are defined likewise and these three points are the vertices of the cevian triangle of tgP. Hence,

Proposition 4. CST *maps the vertices of the cevian triangle of* P *and the vertices of its anticomplement to the vertices of the same cevian triangle, that of* tgP.

2.6. CST *images of some common triangle centers.* Table 1 gives a selection of some CST images. A (6-9-13)-search number is given for each unlisted point in ETC.

Table 1. CST images of some common triangle centers

M	CST(M)	M	CST(M)	M	CST(M)
X_1	X_{10}	X_2	X_{76}	X_3	X_{5562}
X_4	X_4	X_5	4.342332195522807	X_6	X_{39}
X_7	X_{85}	X_8	X_{341}	X_9	5.493555510910763
X_{10}	5.329221045166122	X_{11}	4.196262646186253	X_{12}	2.698123376290196
X_{13}	0.1427165061182335	X_{14}	5.228738830014126	X_{15}	4.707520749612165
X_{16}	-15.70210201702076	X_{17}	2.708683938139388	X_{18}	12.30617330317703

Peter Moses has kindly provided all the pairs $\{M$,CST$(M)\}$ = $\{X_i, X_j\}$ in the ETC (up to X_{5573}) for these {i , j}. Apart from those listed in Table 1 above and excluding X_2 and all the points on the Steiner ellipse for which CST(M) is X_{76}, he has found

i	66	69	100	101	110	513	651	879	925
j	2353	3926	8	3730	3	764	348	5489	847
i	1113	1114	1379	1380	1576	3952			
j	4	4	3557	3558	3202	1089			

3. CST images of lines

Let $\mathcal{L}$ be the line with equation $px + qy + rz = 0$ and trilinear pole $Q = (qr : rp : pq)$.

3.1. *The general case.* In general, the CST image of $\mathcal{L}$ is a nodal cubic with node X_{76} which is tangent to the sidelines of ABC at the traces A_2, B_2, C_2 of $\mathcal{L}(tgQ)$ and meeting these lines again at the traces A_1, B_1, C_1 of $\mathcal{L}(tgtatQ)$.

Indeed, if $\mathcal{L}$ meets BC at $U = (0 : r : -q)$ and G_bG_c at $U' = (q - r : -p : p)$ then $\text{CST}(U) = A_2 = (0 : c^2r : -b^2q)$ and $\text{CST}(U') = A_1 = (0 : c^2(p+q-r) : -b^2(p - q + r))$.

Note that the CST image of the infinite point of $\mathcal{L}$ is the point

$$\left(\frac{(q-r)^3}{a^2} : \frac{(r-p)^3}{b^2} : \frac{(p-q)^3}{c^2}\right)$$

on the cubic. It is also on **K244** as seen below.

The most remarkable example is obtained when $Q = X_2$. Since $\mathcal{L}$ is the line at infinity and since the two trilinear polars coincide into the de Longchamps axis (the isotomic transform of the circumcircle of ABC), we find the cubic **K244** meeting the sidelines of ABC at three inflexion points on the curve (see Figure 2).

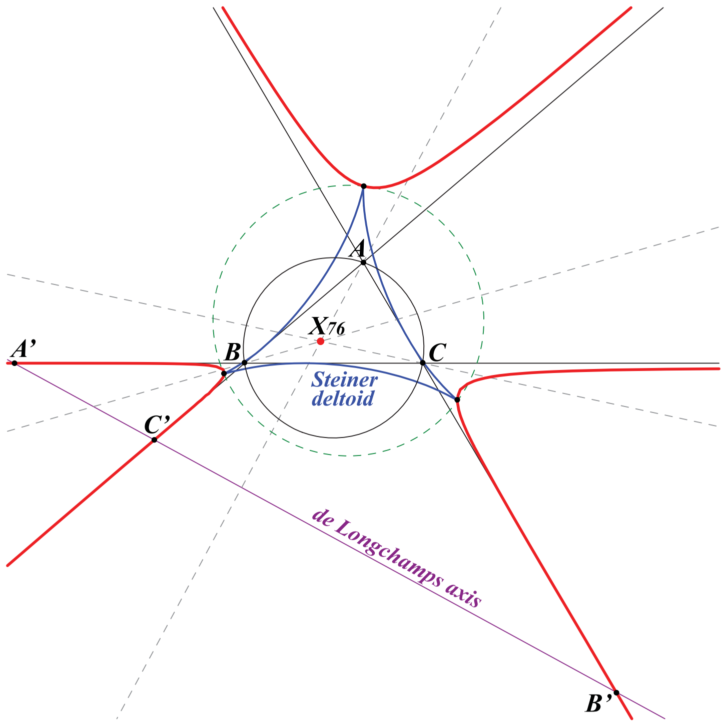

Figure 2. The cubic **K244**

3.2. *Special cases.*

(1) If $\mathcal{L}$ contains X_2 and another point M, the cubic is the line $\mathcal{L}'$ passing through X_{76} and $\mathrm{CST}(M)$ counted three times.

More precisely, if $\mathcal{L}$ meets the Kiepert hyperbola again at E then $\mathcal{L}'$ is the line $X_{76}E$.

(2) If $\mathcal{L}$ is tangent to the Steiner ellipse, the cubic is cuspidal (with cusp X_{76}) and the lines $\mathcal{L}(tgQ)$, $\mathcal{L}(tgtatQ)$ envelope the circum-conic and the in-conic with same perspector X_{76} respectively.

4. CST images of circum-conics

Let $\mathcal{C}(Q)$ be the circum-conic with perspector $Q = (p : q : r) \neq X_2$ (to eliminate the Steiner ellipse case) and equation $pyz + qzx + rxy = 0$.

4.1. *The general case.* In general, the CST image of $\mathcal{C}(Q)$ is a nodal circum-cubic with node N_Q passing through X_{76} which turns out to be a $ps\mathcal{K}$ as in [2]. This cubic has the following properties.

(1) Its pseudo-pivot $P_Q = \left(\dfrac{1}{a^2(-p+q+r)} : \cdots : \cdots\right)$ is $tgtaQ$.

(2) Its pseudo-isopivot $P_Q^* = \left(\dfrac{p^2}{a^2} : \cdots : \cdots\right)$ is tgQ^2.

(3) Its node $N_Q = \left(\dfrac{p}{a^2(-p+q+r)} : \cdots : \cdots\right)$ is $P_Q \times Q$.

(4) Its pseudo-pole $\Omega_Q = \left(\dfrac{p^2}{a^4(-p+q+r)} : \cdots : \cdots\right)$ is $P_Q \times P_Q^*$ or $N_Q \times tgQ$, This node is obtained when the intersections of $C(Q)$ with the line through its center and X_2 are transformed under CST.

(5) The isoconjugate X_{76}^* of X_{76} is $Q \times N_Q = \left(\dfrac{p^2}{a^2(-p+q+r)} : \cdots : \cdots\right)$, obviously on the cubic.

The most remarkable example is obtained when $Q = X_6$ since $\mathcal{C}(Q)$ is the circumcircle (O) of ABC. In this case we find the (third) Musselman cubic **K028**, a stelloid which is $ps\mathcal{K}(X_4, X_{264}, X_3)$. See details in [1] and Figure 3.

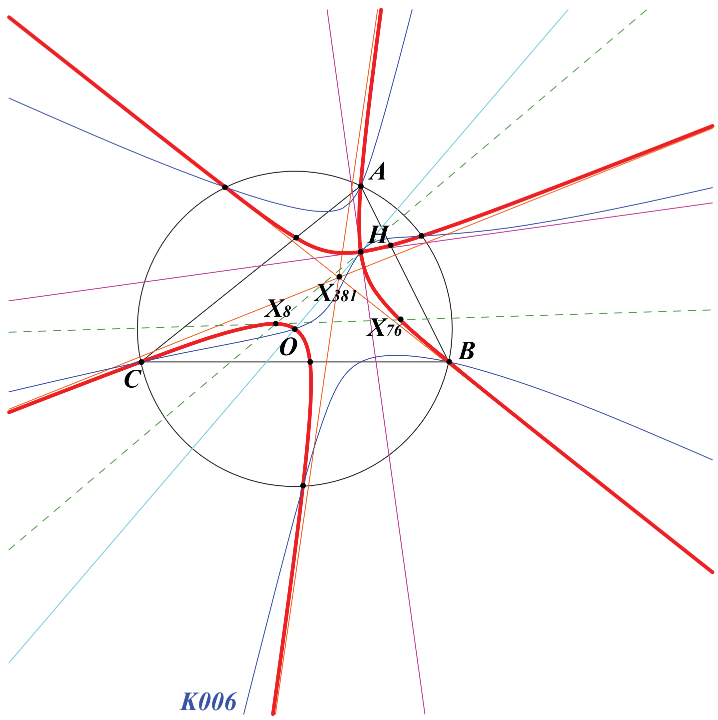

Figure 3. The cubic **K028**

4.2. *Special cases.* The CST image of $\mathcal{C}(Q)$ is a cuspidal circum-cubic if and only if Q lies on two cubics which are the complement of **K196** (the isotomic transform of **K024** with no remarkable center on it) and **K219** (the complement of **K015**) containing X_2, X_{1645}, X_{1646}, X_{1647}, X_{1648}, X_{1649}, X_{1650}. In this latter case, the cusp lies on **K244**.

Figure 4 shows the cubic which is the CST image of $\mathcal{C}(X_{1646})$, a circum-cubic passing through X_{513}, X_{668}, X_{891}, X_{1015}. The cusp is $X_{764} = \text{CST}(X_{513})$. Since X_{891} is a point at infinity, its image also lies on **K244**.

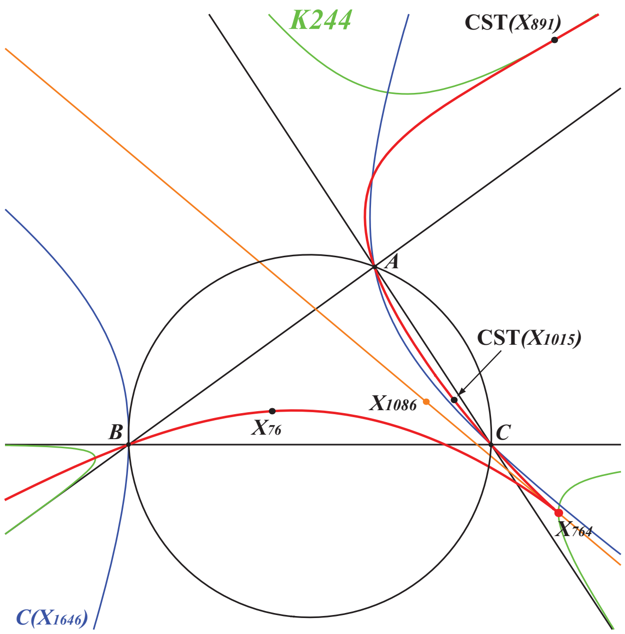

Figure 4. A cuspidal cubic, CST image of $\mathcal{C}(X_{1646})$

4.3. CST *images of some usual circum-conics.* Any $\mathcal{C}(Q)$ which is a rectangular hyperbola must have its perspector Q on the orthic axis, the trilinear polar of X_4. Its CST image $\mathcal{K}(Q)$ is a nodal cubic passing through X_{76} and X_4. Furthermore, its node lies on **K028**, its pseudo-pivot lies on the Steiner ellipse, its pseudo-isopivot lies on the inscribed conic with perspector X_{2052}. The pseudo-pole lies on a complicated quartic.

Table 2 gives a selection of such hyperbolas.

Table 2. CST images of some usual rectangular hyperbolas

Q	$\mathcal{C}(Q)$	$\mathcal{K}(Q)$	N_Q	other centers on $\mathcal{K}(Q)$
X_{523}	Kiepert	$ps\mathcal{K}(X_{850} \times X_{76}, X_{670}, X_{76})$	X_{76}	
X_{647}	Jerabek	$ps\mathcal{K}(X_{520}, X_{99}, X_3)$	X_3	X_{39}, X_{2353}
X_{650}	Feuerbach	$ps\mathcal{K}(X_{4397}, X_{668}, X_4)$	X_8	X_{10}, X_{85}, X_{341}

Remark. When M lies on the Jerabek hyperbola, the points X_3, M and $\text{CST}(M)$ are collinear. This is also true when M lies on the circumcircle.

More generally, for any point N_Q on **K028**, the points M, CST(M), N_Q are collinear if and only if M lies on two circum-conics γ_1, γ_2.

γ_1 is the isogonal conjugate of the parallel δ_1 at X_3 to the line X_4N_Q. γ_1 is obviously a rectangular hyperbola.

γ_2 is the isogonal conjugate of the perpendicular δ_2 at X_3 to the line X_4N_Q. The perspector of γ_2 lies on the circum-conic passing through X_2, X_6.

Note that δ_2 envelopes the Kiepert parabola and that δ_1, δ_2 meet on the Stammler strophoid **K038**.

The CST images of γ_1, γ_2 are two nodal cubics $ps\mathcal{K}$ with nodes $N_1 = N_Q$, N_2 on the Kiepert hyperbola respectively.

5. CST images of some circum-cubics

5.1. CST *images of the cubics* $n\mathcal{K}_0(P, P)$. If $P = (p : q : r)$, the cubic $n\mathcal{K}_0(P, P)$ has an equation of the form

$$\sum_{\text{cyclic}} \frac{x^2(y+z)}{p} = 0 \iff \sum_{\text{cyclic}} \frac{a^2}{p} \times \frac{x^2(y+z)}{p} = 0,$$

which shows that its CST image is the line $\mathcal{L}(tgP)$.

Recall that $n\mathcal{K}_0(P, P)$ is a member of the class **CL026**. It is a cubic having three asymptotes concurring at X_2.

With $P = X_2, X_6, X_{1989}$ we find the cubics **K016**, **K024**, **K064** whose CST images are the de Longchamps axis, the line at infinity, the perpendicular bisector of OH respectively.

The cubics $n\mathcal{K}_0(X_{112}, X_{112})$, $n\mathcal{K}_0(X_{1576}, X_{1576})$, $n\mathcal{K}_0(X_{32}, X_{32})$ give the Euler line, the Brocard axis, the Lemoine axis.

With $P = gtX_{107}$ we have the cubic whose CST image is the line HK. See Figure 5.

5.2. CST *images of the cubics* $c\mathcal{K}(\#P, P^2) = n\mathcal{K}(P^2, P^2, P)$. If $P = (p : q : r)$, the cubic $c\mathcal{K}(\#P, P^2)$ has an equation of the form

$$\sum_{\text{cyclic}} p^2\, x\, (ry - qz)^2 = 0.$$

It is a nodal cubic with node P. Since it is tangent at A, B, C to the sidelines of the antimedial triangle, its CST image must be a cubic curve with node CST(P). This cubic is tangent to the sidelines of ABC at their intersections with $\mathcal{L}(tgP^2)$ and meets these sidelines again on $\mathcal{L}(tgctP)$.

The most remarkable example is obtained when $P = X_2$ since the CST image of the nodal Tucker cubic **K015** $= c\mathcal{K}(\#X_2, X_2)$ is the cubic **K244**. In this case, the two trilinear polars coincide as already point out above.

We conclude with a summary of interesting CST images.

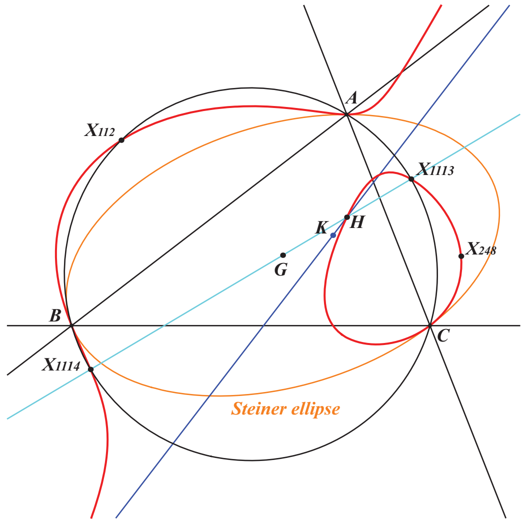

Figure 5. The cubic $n\mathcal{K}_0(P, P)$ with $P = gtX_{107}$

	$\mathcal{C}$	CST($\mathcal{C}$)
Lines	line at infinity	**K244**
	Euler line	line X_4, X_{69}, X_{76}, etc
Conics	Steiner ellipse	X_{76}
	Circumcircle	**K028**
	Kiepert hyperbola	$ps\mathcal{K}(X_{850} \times X_{76}, X_{670}, X_{76})$
	Jerabek hyperbola	$ps\mathcal{K}(X_{520}, X_{99}, X_3)$
	Feuerbach hyperbola	$ps\mathcal{K}(X_{4397}, X_{668}, X_4)$
Cubics	**K024**	line at infinity
	K015	**K244**
	K242	$ps\mathcal{K}(X_{850} \times X_{76}, X_{670}, X_{76})$
Others	**Q066**	Kiepert hyperbola

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