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Asymptotic Directions of Pivotal Isocubics

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Abstract. Given the pivotal isocubic $\mathcal{K} = p\mathcal{K}(\Omega, P)$, we seek all the other isocubics $\mathcal{K}_1 = p\mathcal{K}(\Omega_1, P_1)$ with pole Ω_1 and pivot P_1 which have the same points at infinity, i.e., the same asymptotic directions. We also examine the connection with the Simson lines concurring at a certain given point.

1. Generalities

Recall that the pivotal isocubic $\mathcal{K} = p\mathcal{K}(\Omega, P)$ is the locus of point M such that P, M and the $\Omega-$isoconjugate M^* of M are collinear. With a pole $\Omega(p : q : r)$ and a pivot $P(u : v : w)$, its barycentric equation is

$$\sum_{\text{cyclic}} u\,x(ry^2 - qz^2) = 0 \iff \sum_{\text{cyclic}} p\,yz(wy - vz) = 0 \tag{1}$$

We denote by $\mathcal{F}_{\mathcal{K}}$ the family of all pivotal isocubics having the same points at infinity as $\mathcal{K}$.

Theorem 1 (Pole theorem)**.** *Given $\mathcal{K} = p\mathcal{K}(\Omega, P)$, a pivotal isocubic $\mathcal{K}_1 = p\mathcal{K}(\Omega_1, P_1)$ has the same points at infinity as $\mathcal{K}$ if its pole Ω_1 lies on the cubic $\mathcal{K}_\Omega$ with equation*

$$\sum_{\text{cyclic}} u\,x(y + z - x)(ry - qz) = 0 \tag{2}$$

$$\iff \sum_{\text{cyclic}} pyz(v(x - y + z) - w(x + y - z)) = 0. \tag{3}$$

Theorem 2 (Pivot theorem)**.** *Given $\mathcal{K} = p\mathcal{K}(\Omega, P)$, a pivotal isocubic $\mathcal{K}_1 = p\mathcal{K}(\Omega_1, P_1)$ has the same points at infinity as $\mathcal{K}$ if its pivot P_1 lies on the cubic $\mathcal{K}_P$ with equation*

$$\sum_{\text{cyclic}} u\,(y + z)(ry(x + z) - qz(x + y)) = 0 \tag{4}$$

$$\iff \sum_{\text{cyclic}} p\,(x + y)(x + z)(wy - vz) = 0. \tag{5}$$

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1.1. *Properties of* $\mathcal{K}_\Omega$. $\mathcal{K}_\Omega$ is the pseudo-isocubic $ps\mathcal{K}(\Omega \times cP, G, \Omega)$ where cP is the complement of P and $X_\Omega = \Omega \times cP$ is the barycentric product of Ω and cP. See [3].

$\mathcal{K}_\Omega$ is a circum-cubic passing through Ω and the midpoints M_a, M_b, M_c of BC, CA, AB.

The tangents at A, B, C concur at X_Ω. This point lies on $\mathcal{K}_\Omega$ when G, Ω, P are collinear in which case $\mathcal{K}_\Omega$ is a $p\mathcal{K}$ (see §2).

The equation (2) shows that, for a given Ω, all $\mathcal{K}_\Omega$ form a net of circum-cubics which is generated by three decomposed cubics, one of them being the union of the line BC, the line through the midpoints of AB and AC, the line $A\Omega$, the other two similarly. Generally, this net contains only one circular cubic and only one equilateral cubic.

1.2. *Properties of* $\mathcal{K}_P$. $\mathcal{K}_P$ is the anticomplement of the pseudo-isocubic $\mathcal{K}'_P = ps\mathcal{K}(\Omega \times cP, G, cP)$. See [3].

$\mathcal{K}_P$ is a circum-cubic passing through P and the vertices G_a, G_b, G_c of the antimedial triangle. Moreover, it has the same points at infinity as $\mathcal{K}$ hence, $\mathcal{K}_P$ is a circular cubic (an equilateral cubic) if and only if $\mathcal{K}$ is itself a circular cubic (an equilateral cubic).

The tangents at G_a, G_b, G_c concur at a point which is the anticomplement of X_Ω. This point lies on $\mathcal{K}_P$ when G, Ω, P are collinear in which case $\mathcal{K}_\Omega$ is also a $p\mathcal{K}$ (see §2).

$\mathcal{K}_P$ is itself a pseudo-isocubic if and only if its tangents at A, B, C concur which is realized when G, Ω, P are collinear as above but also when P lies on the circum-conic with center Ω or equivalently when Ω lies on the bicevian conic $\mathcal{C}(G, P)$ with center ccP, the complement of cP. In this latter case, the pseudo-pivot of $\mathcal{K}_P$ lies on the Steiner ellipse and the pseudo-pole lies on $ps\mathcal{K}(ta\Omega, t(G/\Omega), G)$.

The equation (5) shows that, for a given P, all $\mathcal{K}_P$ form another net of circum-cubics which is generated by three decomposed cubics, one of them being the union of the parallels at C, B to AB, AC respectively and the line AP, the other two similarly.

1.3. *A special case.* With $\Omega = P^2$ (barycentric square), $\mathcal{K}_\Omega$ (resp. $\mathcal{K}_P$) is the locus of poles (resp. pivots) of all $p\mathcal{K}$ having asymptotes parallel to the cevian lines of P.

1.4. *Examples.* We show several examples with known cubics for which G, Ω, P are not collinear.

1.4.1. $\mathcal{K}$ *is the McCay cubic.* With $\Omega = K$ and $P = O$, $\mathcal{K}$ is the McCay cubic **K003**. We know that it has three real asymptotes perpendicular to the sidelines of the Morley triangle.

$\mathcal{K}_\Omega$ is **K307** = $ps\mathcal{K}(X_{51}, X_2, X_6)$ with equation :

$$\sum_{\text{cyclic}} a^2 S_A\, x(y+z-x)(c^2y - b^2z) = 0 \tag{6}$$

It passes through K, X_{53}, X_{216}, X_{1249}. The tangents at A, B, C concur at X_{51}, centroid of the orthic triangle. See Figure 1.

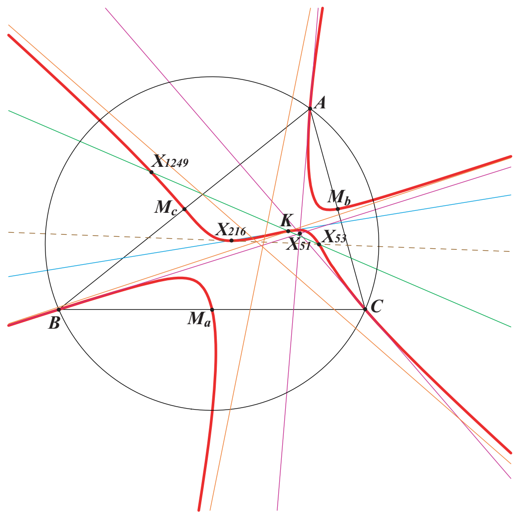

Figure 1. $\mathcal{K}_\Omega$ with $\Omega = K$ and $P = O$

Each point on the curve is the pole of a $p\mathcal{K}$ having asymptotes parallel to those of the McCay cubic. $p\mathcal{K}(X_{53}, X_4)$ = **K049** is the McCay cubic of the orthic triangle, $p\mathcal{K}(X_{216}, X_{20})$ = **K096** contains X_3, X_5, X_{20} and the tangential E_{367} of H in the Darboux cubic.

$\mathcal{K}_P$ is denoted by $\mathcal{K}_O^{++}$ = **K080** : it is a central cubic with center O, having three real asymptotes perpendicular to the sidelines of the Morley triangle and concurring at O. Each point on the curve is the pivot of a $p\mathcal{K}$ having asymptotes parallel to those of the McCay cubic.

Its equation is :

$$\sum_{\text{cyclic}} a^2\,(x+y)(x+z)(c^2 S_C\, y - b^2 S_B\, z) = 0 \qquad (7)$$

$\mathcal{K}_O^{++}$ is a circum-cubic passing through :
– O, H, L, X_{1670}, X_{1671},
– the vertices G_a, G_b, G_c of the antimedial triangle,
– the reflections H_A, H_B, H_C of H in A, B, C,
– the reflections A_O, B_O, C_O of A, B, C in O,
– the points U_a, U_b, U_c which also lie on the Neuberg cubic and on the circle with center L and radius $2R$. These points are the images under $\mathsf{h}(H, 2)$ of the points V_a, V_b, V_c, intersections of the Napoleon cubic with the circumcircle. See Figure 2.

1.4.2. *$\mathcal{K}$ is the orthocubic.* With $\Omega = K$ and $P = H$, $\mathcal{K}$ is the orthocubic **K006**. $\mathcal{K}_\Omega$ is **K260**, a nodal cubic with node K and passes through X_{69}, X_{206}, X_{219},

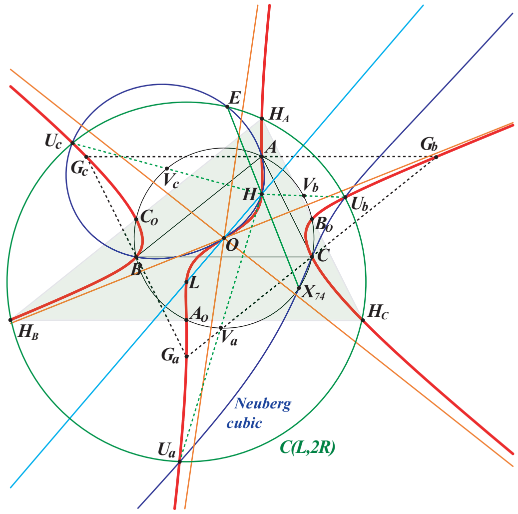

Figure 2. $\mathcal{K}_O^{++}$ and the Neuberg cubic

X_{478}, X_{577}, X_{1249}, X_{2165}. The tangents at A, B, C concur at X_{184}. Its equation is :

$$\sum_{\text{cyclic}} a^4 S_A(y-z)yz - (b^2-c^2)(c^2-a^2)(a^2-b^2)\, xyz = 0 \tag{8}$$

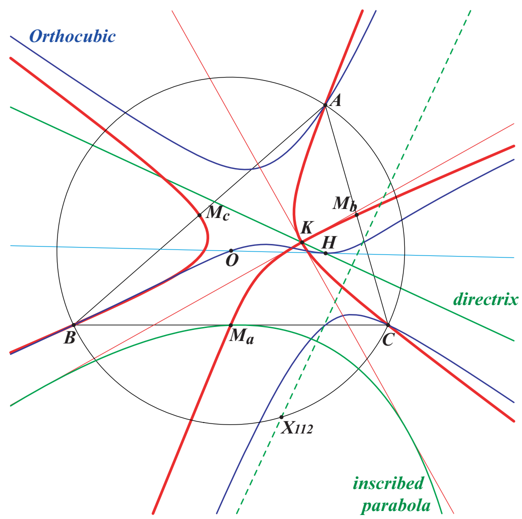

Figure 3. $\mathcal{K}_\Omega$ with $\Omega = K$ and $P = H$

An easy construction of $\mathcal{K}_\Omega$ is the following : the trilinear polar q of any point Q on the Euler line meets the lines KM_a, KM_b, KM_c at Q_a, Q_b, Q_c. The triangles

ABC and $Q_aQ_bQ_c$ are perspective at M and the locus of M is $\mathcal{K}_\Omega$. Furthermore, q envelopes the inscribed parabola with focus X_{112} and directrix the line HK. See Figure 3.

$\mathcal{K}_P$ is **K617**, the anticomplement of **K009**. It is also a nodal cubic with node H and it passes through X_{20}, X_{68}, X_{254}, X_{315}, X_{2996}. The construction seen above for $\mathcal{K}_\Omega$ is easily adapted for $\mathcal{K}_P$: it is enough to replace the Euler line by the line X_2X_{216} and K by H. Similarly we obtain another inscribed parabola with focus X_{107} and directrix the line X_4X_{51}. See Figure 4. The equation of $\mathcal{K}_P$ is :

$$\sum_{\text{cyclic}} a^2\,(x+y)(x+z)(y/S_C - z/S_B) = 0 \tag{9}$$

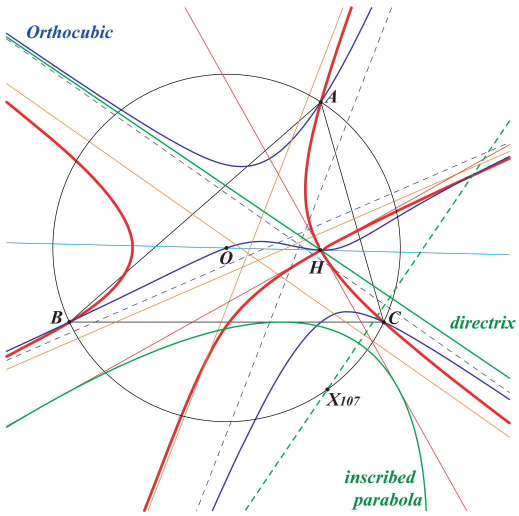

Figure 4. $\mathcal{K}_P$ with $\Omega = K$ and $P = H$

In the two cubics, the tangents at the nodes are parallel to the asymptotes of the Jerabek hyperbola.

From all this, we see that each point Q on the Euler line gives a pole ω on $\mathcal{K}_\Omega$ and a corresponding pivot π on $\mathcal{K}_P$ such that the cubic $p\mathcal{K}(\omega, \pi)$ has its asymptotes parallel to those of the orthocubic. For example, with $Q = G$, $Q = O$ and $Q = H$, we find $p\mathcal{K}(X_{69}, X_{315})$, $p\mathcal{K}(X_{577}, X_{20})$ and $p\mathcal{K}(X_{2165}, X_{68})$ respectively.

1.4.3. *A selection of cubics* $\mathcal{K}_\Omega$, $\mathcal{K}_P$ *and* $\mathcal{K}'_P$. Table 1 below shows a small selection of these cubics, specially those connected with the usual and well known cubics $\mathcal{K}$ in triangle geometry. When a cubic is not listed in [2], a list of centers is provided.

Some of these examples are detailed below.

Table 1. $\mathcal{K}$ and the related cubics $\mathcal{K}_\Omega$, $\mathcal{K}_P$, $\mathcal{K}'_P$

$\mathcal{K}$	$\mathcal{K}_\Omega$	$\mathcal{K}_P$	$\mathcal{K}'_P$
K001	X_6, X_{1249}, X_{1989}, X_{1990}, X_{3163}	**K449**	**K446**
K002	**K002**	**K007**	**K002**
K003	**K307**	**K080**	**K026**
K004	X_6, X_{393}, X_{1249}	X_4, X_{20}	**K376**
K005	X_6, X_{233}, X_{1249}, X_{2963}	X_4, X_5, X_{20}, X_{2888}	**K569**
K006	**K260**	**K617**	**K009**
K034	**K345**	**K034**	**K345**

2. Pivotal $\mathcal{K}_\Omega$ and $\mathcal{K}_P$

2.1. *Theorem and corollaries.*

Theorem 3. *$\mathcal{K}_\Omega$ and $\mathcal{K}_P$ are pivotal isocubics if and only if*

$$(q-r)u + (r-p)v + (p-q)w = 0$$

i.e. if and only if G, Ω, P are collinear.

Corollary 4. *$\mathcal{K}_\Omega$ has pivot G and pole ω, the barycentric product of Ω and the complement of P. This point is the common tangential of A, B, C.*

This pole lies on the line through Ω, the barycentric square of Ω, the complement of the isotomic conjugate of Ω.

$\mathcal{K}_\Omega$ contains the following points :

- A, B, C, G
- M_a, M_b, M_c
- Ω, ω, cP (complement of P), P^* ($\Omega-$isoconjugate of P)
- the isotomic conjugate of the anticomplement of ω and its complement

Corollary 5. *$\mathcal{K}_P$ is an isotomic pivotal isocubic with pivot the anticomplement of ω. This point is the common tangential of G_a, G_b, G_c.*

This pivot lies on the line through the anticomplement of Ω, the isotomic conjugate of Ω, the anticomplement of the barycentric square of Ω.

$\mathcal{K}_P$ contains the following points :

- A, B, C, G
- G_a, G_b, G_c
- P, the anticomplements of Ω, ω, P^*
- the isotomic conjugate of the anticomplement of ω

Corollary 6. *$\mathcal{K}_\Omega$ is the complement of $\mathcal{K}_P$. The two cubics are tangent at G to the line $G\omega$.*

Corollary 7. *$\mathcal{K}_\Omega$, $\mathcal{K}_P$ have the same points at infinity which are those of $\mathcal{K}$.*

From all this, we notice that $\mathcal{K}_\Omega$ and $\mathcal{K}_P$ have nine common points : A, B, C, G (double), the three points at infinity of $\mathcal{K}$, the isotomic conjugate of the anticomplement of ω.

When we choose Ω or P at G, we obtain :

Corollary 8. (1) *Any isotomic $p\mathcal{K}$ is the locus of pivots of all $p\mathcal{K}$ having the same asymptotic directions as itself.*

(2) *Any $p\mathcal{K}$ with pivot G is the locus of poles of all $p\mathcal{K}$ having the same asymptotic directions as itself.*

See the three examples below.

A line ℓ_G through G meets $\mathcal{K}_\Omega$ at two points Ω_1 and Ω_2 which are $\omega-$isoconjugate and collinear with G.

Denote by P_1, P_2 the anticomplements of Ω_2, Ω_1 respectively (reverse order). These points lie on $\mathcal{K}_P$.

Theorem 9. *The two pivotal isocubics $\mathcal{K}_1 = p\mathcal{K}(\Omega_1, P_1)$ and $\mathcal{K}_2 = p\mathcal{K}(\Omega_2, P_2)$ have the same points at infinity as $\mathcal{K} = p\mathcal{K}(\Omega, P)$*

Construction : given Ω and P collinear with G, let Γ_ℓ be the inscribed conic in ABC which is tangent at G to ℓ_G. The trilinear pole Q_ℓ of ℓ_G lies on the Steiner circum-ellipse, and the trilinear pole of the tangent at Q_ℓ to the Steiner circum-ellipse is the perspector of Γ_ℓ.

Now draw the two (not necessarily real) tangents to Γ_ℓ passing through ω. These tangents meet ℓ_G at Ω_2, Ω_1. The construction of P_1, P_2 follows easily.

Beware P_1, P_2 are two points on $\mathcal{K}_P$ but are not isotomic conjugates on this cubic.

2.2. *Examples.*

2.2.1. *$\mathcal{K}$ is the Thomson cubic.* When $\mathcal{K}$ is the Thomson cubic **K002** with $\Omega = K$ (Lemoine point) and $P = G$ (centroid), $\mathcal{K}_\Omega$ is the Thomson cubic again and $\mathcal{K}_P$ is the Lucas cubic **K007**. In other words, all the corresponding cubics $\mathcal{K}_1 = p\mathcal{K}(\Omega_1, P_1)$ and $\mathcal{K}_2 = p\mathcal{K}(\Omega_2, P_2)$ have the same points at infinity which are those of the Thomson and Lucas cubics.

The following table shows several examples of corresponding points Ω_1, P_1 and Ω_2, P_2.

Ω_1	P_1	cubic or X_i for $i =$	Ω_2	P_2	cubic or X_i for $i =$
X_1	X_8	**K308**	X_1	X_8	**K308**
X_2	X_{69}	**K007**	X_6	X_2	**K002**
X_3	X_{20}	2,3,20,1032,1073,1498	X_4	X_4	**K181**
X_9	X_{329}	2,9,188,282,329,1034,1490	X_{57}	X_7	1,2,7,57,145,174,1488,2089
X_{223}	E_{623}		X_{282}	X_{189}	2,8,9,84,189,282
X_{1073}	X_{253}	2,3,64,69,253,1073,3146	X_{1249}	E_{624}	
E_{630}	?		E_{668}	X_{1034}	
E_{382}	E_{625}		E_{553}	X_{1032}	

Remark. E_{623} is the anticomplement of X_{282}, E_{624} is the anticomplement of X_{1073}, E_{382} is the complement of X_{1032}, E_{630} is the complement of X_{1034}.

2.2.2. *$\mathcal{K}$ is the Grebe cubic.* When $\mathcal{K}$ is the Grebe cubic **K102** with $\Omega = P = K$, we obtain $\mathcal{K}_\Omega = p\mathcal{K}(X_{39}, X_2)$ and $\mathcal{K}_P = p\mathcal{K}(X_2, X_{76})$ = **K141**. We find six cubics with the same points at infinity :

Ω_1	P_1	cubic or X_i for $i =$	Ω_2	P_2	cubic or X_i for $i =$
X_3	X_{22}	2,3,22,159	X_{427}	X_4	2,4,76,141,193,427,1843
X_6	X_6	**K102**	X_{141}	X_{69}	2,20,69,141,427
X_{39}	X_2	2,3,6,39,141,427	X_2	X_{76}	**K141**

2.2.3. *$\mathcal{K}$ is an isogonal circular cubic.* The points at infinity of $\mathcal{K}$ need not be real. For example, with $\Omega = K$ and $P = X_{524}$ (infinite point of the line GK), $\mathcal{K}$ is now a circular cubic passing through G, K, X_{111} (Parry point), with singular focus X_{1296} (antipode of the Parry point on the circumcircle). The real asymptote is the parallel to GK at X_{111}.

In this case, $\mathcal{K}_\Omega = p\mathcal{K}(X_{187}, X_2)$ = **K043** (Droussent medial cubic) and $\mathcal{K}_P = p\mathcal{K}(X_2, X_{316})$ = **K008** (Droussent cubic). All the cubics defined in the following table are circular with a real infinite point X_{524}.

Ω_1	P_1	cubic or X_i for $i =$	Ω_2	P_2	cubic or X_i for $i =$
X_3	X_{858}	2,3,66,524,858,895	X_{468}	X_4	**K209**
X_6	X_{524}	1,2,6,111,524,2930	X_{524}	X_{69}	2,20,69,468,524,2373
X_{67}	X_{67}	**K103**	E_{406}	E_{618}	
X_{111}	X_{671}	**K273**	X_{2482}	E_{620}	
X_{187}	X_2	**K043**	X_2	X_{316}	**K008**

Remark. E_{618} is the anticomplement of X_{67}, E_{620} is the anticomplement of X_{111}, E_{406} is the midpoint of X_6, X_{110}.

3. Isogonal and isotomic cubics of the pencil $\mathcal{F}_\mathcal{K}$

We suppose now that $\mathcal{K} = p\mathcal{K}(\Omega, P)$ is neither an isogonal nor an isotomic isocubic, i.e., that Ω is distinct of $K = X_6$ and $G = X_2$.

3.1. *Theorem and consequences.*

Theorem 10. (1) *$\mathcal{F}_\mathcal{K}$ contains one isogonal isocubic if and only if the pivot P of $\mathcal{K}$ lies on the line $\mathcal{L}_g$ passing through H and the $\Omega-$isoconjugate of O.*

(2) *$\mathcal{F}_\mathcal{K}$ contains one isotomic isocubic if and only if the pivot P of $\mathcal{K}$ lies on the line $\mathcal{L}_t$ passing through G and Ω.*

Remark. These two lines are always perfectly defined since Ω is neither K nor G. They coincide if and only if $\Omega = O$: for any $p\mathcal{K}(O, P)$ with P on the Euler line, there is one isogonal isocubic and one isotomic isocubic having asymptotes parallel to those of $p\mathcal{K}(O, P)$.

Corollary 11. *$\mathcal{F}_\mathcal{K}$ contains one isogonal isocubic and one isotomic isocubic if and only if the pivot P of $\mathcal{K}$ is the intersection Q of $\mathcal{L}_g$ and $\mathcal{L}_t$.*

Remark. $\mathcal{L}_g$ and $\mathcal{L}_t$ are parallel if and only if Ω lies on the rectangular hyperbola $\mathcal{H}$ passing through G, O, K, X_{110} (focus of the Kiepert parabola) and also X_{154},

X_{354}, X_{392}, X_{1201}, X_{2574}, X_{2575}. Its asymptotes are parallel to those of the Jerabek hyperbola.

The intersections (other than X_{110}) with the circumcircle are three points on the Thomson cubic which are the vertices of the Thomson triangle. The center of $\mathcal{H}$ is the midpoint of GX_{110}. See Figure 5. The equation of this hyperbola is :

$$\sum_{\text{cyclic}} (b^2 - c^2)(b^2c^2x^2 + a^2 S_A\, yz) = 0$$

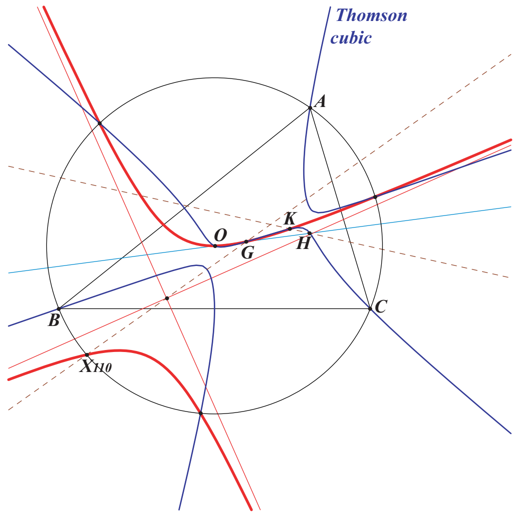

Figure 5. The rectangular hyperbola $\mathcal{H}$

3.2. *Examples.*

- With $\Omega = I$ (incenter), we find $Q = X_8$ (Nagel point). Hence, $p\mathcal{K}(X_1, X_8)$ = **K308** generates a family $\mathcal{F}_{\mathcal{K}}$ of isocubics containing one isogonal isocubic (the Thomson cubic **K002**) and one isotomic isocubic (the Lucas cubic **K007**).
- With $\Omega = O$ and $P = G$, $\mathcal{K}$ = **K168**. The isogonal cubic is $p\mathcal{K}(K, X_{193})$ and the isotomic cubic is $p\mathcal{K}(G, H)$ = **K170**. See Figure 6.

4. Circular $\mathcal{K}_\Omega$ cubics

We have seen that $\mathcal{K}_P$ is circular if and only if $\mathcal{K}$ is itself circular. We have the following theorems for $\mathcal{K}_\Omega$.

Theorem 12. (1) *For any pole Ω distinct of O, there is one and only one circular $\mathcal{K}_\Omega$ which passes through O and Ω.*

(2) *When $\Omega = O$, there are infinitely many circular $\mathcal{K}_\Omega$ forming a pencil of cubics passing through O. In this case, P must lie on the de Longchamps axis.*

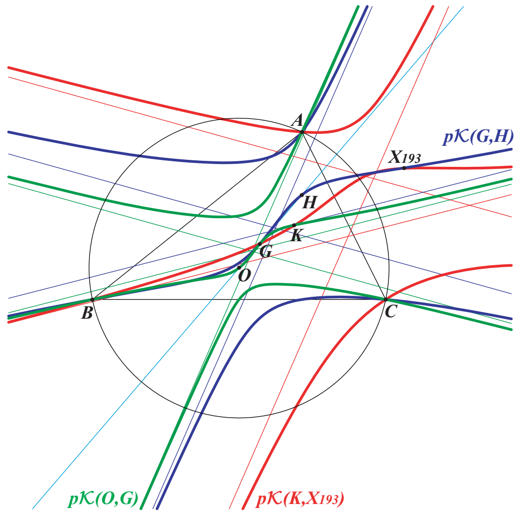

Figure 6. $p\mathcal{K}(O,G)$, $p\mathcal{K}(K,X_{193})$ and $p\mathcal{K}(G,H)$

For example, with $P = X_{858}$, $\mathcal{K}_\Omega$ is the Droussent medial cubic **K043**.

Theorem 13. (1) *For any pivot P distinct of X_{69}, there is one and only one circular $\mathcal{K}_\Omega$.*

(2) *When $P = X_{69}$, there are infinitely many circular $\mathcal{K}_\Omega$ forming a pencil of cubics passing through O and Ω which must lie on the line at infinity. The isogonal conjugate of Ω lies on the cubic (and on the circumcircle).*

The table gives a selection of such cubics.

Ω	centers on the cubic	cubic
X_{30}	$X_3, X_4, X_{30}, X_{74}, X_{133}, X_{1511}$	**K446**
X_{519}	$X_1, X_3, X_{106}, X_{214}, X_{519}, X_{1319}$	
X_{524}	$X_2, X_3, X_6, X_{67}, X_{111}, X_{187}, X_{468}, X_{524}, X_{1560}, X_{2482}$	**K043**
X_{527}	$X_3, X_9, X_{57}, X_{527}, X_{1155}, X_{2291}$	
X_{532}	$X_3, X_{13}, X_{16}, X_{532}, X_{618}, X_{2380}$	
X_{533}	$X_3, X_{14}, X_{15}, X_{533}, X_{619}, X_{2381}$	
X_{758}	$X_3, X_{10}, X_{36}, X_{65}, X_{758}, X_{759}$	
X_{2393}	$X_3, X_{25}, X_{206}, X_{858}, X_{2373}, X_{2393}$	

5. $p\mathcal{K}$ with three real asymptotes

Let $\mathcal{K} = p\mathcal{K}(\Omega, P)$ be the pivotal isocubic with pole $\Omega(p : q : r)$ and pivot $P(u : v : w)$. Let $\mathcal{C}_\Omega$ be the circum-conic with perspector Ω and center $O_\Omega = p(q + r - p) : q(r + p - q) : r(p + q - r)$, the $G-$Ceva conjugate of Ω i.e. the perspector of the medial triangle and the anticevian triangle of Ω. Let Γ_Ω be the homothetic of $\mathcal{C}_\Omega$ under $\mathsf{h}(O_\Omega, 3)$.

Theorem 14. *$p\mathcal{K}$ has three real asymptotes if and only if P lies inside* [1] *a tricuspidal quartic $\mathcal{Q}_\Omega$ tritangent to $\mathcal{C}_\Omega$ and having its cusps on Γ_Ω.*

Remark. $\mathcal{Q}_\Omega$ is bitangent to the line at infinity at the two points where $\mathcal{C}_\Omega$ meets the line at infinity.

Corollary 15 (isogonal $p\mathcal{K}$)**.** *From this remark, it is clear that $\mathcal{Q}_\Omega$ is bicircular if and only if $\Omega = K$. In this case, $O_\Omega = O$, $\mathcal{C}_\Omega$ is the circumcircle and $\mathcal{Q}_\Omega$ is a deltoid, the envelope of axes of inscribed parabolas. This result is already mentioned in* [1].

Corollary 16 (isotomic $p\mathcal{K}$)**.** *$\mathcal{Q}_\Omega$ contains A, B, C if and only if $\Omega = G$. In this case, $O_\Omega = G$, $\mathcal{C}_\Omega$ is the Steiner ellipse.*

$\mathcal{Q}_G$ has three cusps lying on the medians of ABC and on Γ_Ω, the homothetic of the Steiner ellipse under $\mathsf{h}(G, 3)$. It is tangent at A, B, C to the Steiner ellipse. See Figure 7.

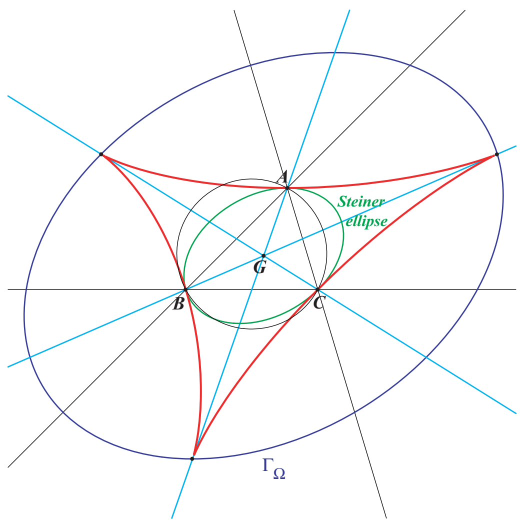

Figure 7. The tricuspidal quartic $\mathcal{Q}_G$

The equation of $\mathcal{Q}_G$ is :

$$32 \sum_{\text{cyclic}} x\,(y^3 + z^3) + 61 \sum_{\text{cyclic}} y^2 z^2 + 118 \sum_{\text{cyclic}} x^2 yz = 0 \qquad (10)$$

Remark. When Ω lies on the inscribed Steiner ellipse, $\mathcal{Q}_\Omega$ decomposes into the line at infinity counted twice and a parabola. In this case, one of the fixed points of the isoconjugation lies at infinity.

[1] P is said to be inside the quartic when it lies in the same region of the plane as O_Ω.

6. Asymptotes and Simson lines

Let $\mathcal{K} = p\mathcal{K}(\Omega, P)$ be the pivotal isocubic with pole $\Omega(p : q : r)$ and pivot $P(u : v : w)$ and let M be a point. It is known that there are three (real or not, distinct or not) Simson lines that pass through M since the envelope of all Simson lines is the well known Steiner deltoid $\mathcal{H}_3$, a tricuspidal bicircular quartic of class 3.

6.1. *Construction of the Simson lines passing through M.* Jean-Pierre Ehrmann has found a simple conic construction of these lines which is as follows. Draw the rectangular circum-hyperbola $\mathcal{H}_M$ passing through M and its image $\mathcal{H}'_M$ under the translation that maps H onto M. $\mathcal{H}'_M$ meets the circumcircle of ABC at four (real or not) points. One of them (always real) is the reflection of H about the center of $\mathcal{H}_M$ and this point also lies on $\mathcal{H}_M$. The three other points (one is always real) are those whose Simson lines pass through M.

Recall that these three points are all real when M lies inside the Steiner deltoid $\mathcal{H}_3$.

6.2. *Concurrent Simson lines and cevian lines.* Three Simson lines concurring at Q are parallel to the cevian lines of a certain point M if and only if M lies on the McCay cubic **K003**. If $M = (\alpha : \beta : \gamma)$, this point Q is given by $Q = (b^2c^2\alpha^2(\beta + \gamma) ::)$. Hence it is the barycentric product $tgM \times ctM$. The mapping $M \mapsto Q$ has numerous properties we shall not consider in this paper. See [5].

For example, if we take $M = X_3 = O$, we obtain a point Q which is X_{5562} in ETC. Its first barycentric coordinate is : $a^2S_A^2\left[a^2(b^2+c^2) - (b^2-c^2)^2\right]$ and its SEARCH number is $1.84961021841713\cdots$. This point is the intersection of many lines such as X_2X_{389}, X_3X_{49}, X_4X_{69}, X_5X_{51}, etc.

Figure 8 shows these Simson lines and the corresponding hyperbolas $\mathcal{H}_O$ (here the Jerabek hyperbola), $\mathcal{H}'_O$ meeting the circumcircle at X_{74} and three points Q_1, Q_2, Q_3 whose Simson lines concur at $Q = X_{5562}$.

6.3. *Theorems.* In this section, we characterize the cubics $\mathcal{K}$ for which the asymptotes are parallel to three Simson lines concurring at a certain point $M(\alpha : \beta : \gamma)$. The equation of these three Simson lines is given by

$$\sum_{\text{cyclic}} a^2 \Big(\underbrace{\alpha(y-z) - x(\beta-\gamma)}_{A_1}\Big)\Big(\underbrace{\beta(x+z) - y(\alpha+\gamma)}_{A_2}\Big)\Big(\underbrace{\gamma(x+y) - z(\alpha+\beta)}_{A_3}\Big) = 0,$$

which is then more simply rewritten under the form :

$$\sum_{\text{cyclic}} a^2 A_1A_2A_3 = 0.$$

In this case, the equation of the parallels at M to the asymptotes of $\mathcal{K}$ is :

$$\sum_{\text{cyclic}} p\ (w\, A_2 - v\, A_3)A_2A_3 = 0.$$

When we express that these two equations have the same solutions when $(x : y : z)$ is a point at infinity, we obtain three conditions which are linear with respect

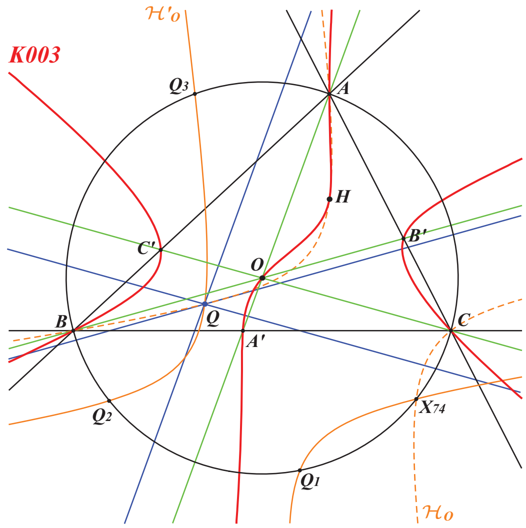

Figure 8. Concurrent Simson lines and cevian lines

to all the variables namely $(p : q : r)$, $(u : v : w)$ and $(\alpha : \beta : \gamma)$. In other words, if two of the three points Ω, P, M are chosen, the coordinates of the third point are given by a system of three linear equations with a corresponding 3×3 matrix generally of rank 3.

Since the system has always the trivial and improper solution $(0 : 0 : 0)$, we will find at least one proper solution if and only if the determinant of each of the three matrices above is zero. This gives three conditions involving two of the three points Ω, P, M. These conditions are

$$(\Omega P) \ : \ \sum_{\text{cyclic}} u\, a^2 S_A(c^2 q - b^2 r) = 0 \iff \sum_{\text{cyclic}} p\, b^2 c^2 (S_B\, v - S_C\, w) = 0, \tag{11}$$

$$\begin{aligned} (\Omega M) \ &: \ \sum_{\text{cyclic}} \alpha\, qr \Big((b^2 - c^2)p - a^2(q - r) \Big) = 0 \\ &\iff \sum_{\text{cyclic}} a^2 qr \Big(\alpha\,(q - r) + p\,(\beta - \gamma) \Big) = 0, \end{aligned} \tag{12}$$

$$\begin{aligned} (PM) \ &: \ \sum_{\text{cyclic}} \alpha (u + v)(u + w)(S_B\, v - S_C\, w) = 0 \\ &\iff \sum_{\text{cyclic}} a^2 (u + v)(u + w) \Big((\alpha + \beta - \gamma)\, v - (\alpha - \beta + \gamma)\, w \Big) = 0, \end{aligned} \tag{13}$$

Remarks. (1) (ΩP) is linear in Ω and P, (ΩM) and (PM) are linear in M.

(2) (ΩP) identically vanishes when $\Omega = X_6$ or $P = X_4$.

(3) (ΩM) identically vanishes when $\Omega = X_6$ (or when Ω is a midpoint of ABC not giving a proper cubic).

(4) (PM) identically vanishes when $P = X_4$ (or when P is a vertex of the antimedial triangle not giving a proper cubic).

(5) When $P = X_{69}$, the conditions (ΩP) and (PM) show that the points Ω and M must lie on the line $GK = X_2X_6$.

Each time a condition identically vanishes, the system above has one and only one solution since the rank of at least one of the matrices above is 2. This is examined in the next section.

6.4. *Special cases.*

6.4.1. $\Omega = X_6$. When $\Omega = X_6$, the cubic $\mathcal{K}$ is a pivotal isogonal cubic and its asymptotes are parallel to the Simson lines that pass through $M = cP$, the complement of the pivot.

For example, with $P = X_3$ we have $M = cP = X_5$: the asymptotes of the McCay cubic **K003** are the parallels at G to the Simson lines passing through X_5 which are in fact the axes of the Steiner deltoid.

6.4.2. $P = X_4$. When $P = X_4$, the cubic $\mathcal{K}$ is a pivotal isogonal cubic with respect to the orthic triangle. Its asymptotes are parallel to the Simson lines that pass through $M = \Omega \times X_{69}$ (barycentric product). Conversely, if M is given, the pole Ω is that of the isoconjugation that swaps the orthocenter $H = X_4$ and M.

For example, with $M = X_5$ we have $\Omega = X_4 \times X_5 = X_{53}$: the corresponding cubic is the McCay orthic cubic **K049** whose asymptotes are the parallels at X_{51} to the Simson lines passing through X_5 as above.

6.4.3. $\Omega = X_6$ *and* $P = X_4$. This gives the orthocubic **K006** whose asymptotes are parallel to the Simson lines passing through $O = X_3$. In this case, these asymptotes are not concurrent.

6.5. *Interpretation of the three conditions in the general case.*

6.5.1. *The linear conditions.* The conditions (11), (11), (13) above represent actually six equations and four of them are linear with respect to the coordinates of at least one of the three points Ω, P, M. These four equations give the following propositions.

Proposition 17. *For a given pole $\Omega \neq X_6$, the asymptotes of $\mathcal{K}$ are parallel to the Simson lines of a certain point M if and only if its pivot P lies on the line $\mathcal{L}(\Omega, P)$ with equation*

$$\sum_{\text{cyclic}} a^2 S_A (c^2 q - b^2 r) x = 0.$$

In this case, this point M must lie on the line $\mathcal{L}(\Omega, M)$ with equation

$$\sum_{\text{cyclic}} qr\big((b^2 - c^2)p - a^2(q - r)\big)x = 0.$$

$\mathcal{L}(\Omega, P)$ is the Steiner line of the isogonal conjugate of the infinite point of the trilinear polar of the isogonal conjugate of Ω and therefore passes through X_4.

If Γ_Ω is the circum-conic with perspector Ω, the line $\mathcal{L}(\Omega, M)$ is the conjugated diameter with respect to Γ_Ω of the trilinear polar of the isotomic conjugate of the isogonal conjugate of Ω. Obviously, this line contains the center of Γ_Ω.

Proposition 18. *For a given pivot $P \neq X_4$, the asymptotes of $\mathcal{K}$ are parallel to the Simson lines of a certain point M if and only if its pole Ω lies on the line $\mathcal{L}(P, \Omega)$ with equation*

$$\sum_{\text{cyclic}} b^2c^2(S_B\,v - S_C\,w)x = 0.$$

In this case, this point M must lie on the line $\mathcal{L}(P, M)$ with equation

$$\sum_{\text{cyclic}} (u+v)(u+w)(S_B\,v - S_C\,w)x = 0.$$

$\mathcal{L}(P, \Omega)$ is the trilinear polar of the isogonal conjugate of the infinite point of the trilinear polar of the barycentric quotient $X_4 \div P$ or equivalently the X_4-isoconjugate of P.

$\mathcal{L}(P, M)$ contains cP. It is the trilinear polar of the isoconjugate of the infinite point of the trilinear polar of the barycentric quotient $X_4 \div P$ under the isoconjugation with pole cP.

For example,

- $\mathcal{L}(\Omega = X_2, P)$ is the line through X_4, X_{69}, X_{76}, etc, and $\mathcal{L}(\Omega, P = X_2)$ is the Brocard axis,
- $\mathcal{L}(\Omega = X_2, M)$ and $\mathcal{L}(P = X_2, M)$ coincide into the line X_2, X_6, X_{69}, etc.

6.5.2. *The other conditions.* The remaining two equations are of degree 3 and lead to two cubic curves. They correspond to the choice of a given point M whose isogonal conjugate of isotomic conjugate is denoted gtM, also the barycentric product $M \times X_6$.

Property 1. *For a given point M, there is a cubic $\mathcal{K}$ whose asymptotes are parallel to the Simson lines passing through M if and only if its pole Ω lies on the cubic $\mathcal{K}(M, \Omega)$ which is $ps\mathcal{K}(gtM, X_2, X_6) = ps\mathcal{K}(M \times X_6, X_2, X_6)$ with equation*

$$\sum_{\text{cyclic}} a^2 \Big[\alpha\,(y-z) + x\,(\beta-\gamma)\Big] yz = 0.$$

Property 2. *For a given point M, there is a cubic $\mathcal{K}$ whose asymptotes are parallel to the Simson lines passing through M if and only if its pivot P lies on the cubic $\mathcal{K}(M, P)$ which is the anticomplement of $ps\mathcal{K}(gtM, X_2, X_4) = ps\mathcal{K}(M \times X_6, X_2, X_4)$. The equation of $\mathcal{K}(M, P)$ is*

$$\sum_{\text{cyclic}} a^2 \Big[(\alpha+\beta-\gamma)\,y - (\alpha-\beta+\gamma)\,z\Big](x+y)(x+z) = 0$$

and that of its complement is very similar to the equation above namely

$$\sum_{\text{cyclic}} a^2 \left[\alpha\,(y-z) - x\,(\beta-\gamma)\right] yz = 0.$$

Example 1 : When we consider the cubics having their asymptotes parallel to the Simson lines passing through the circumcenter $O = X_3$, we find

- $\mathcal{K}(M = X_3, \Omega)$ = **K260** = $ps\mathcal{K}(X_{184}, X_2, X_6)$ passing through X_6, X_{69}, X_{206}, X_{219}, X_{478}, X_{577}, X_{1249}, X_{2165}.
- $\mathcal{K}(M = X_3, P)$ passing through X_4, X_{20}, X_{68}, X_{254}, X_{315}, X_{2996}. It is the anticomplement of $ps\mathcal{K}(X_{184}, X_2, X_4)$ which contains X_3, X_4, X_{32}, X_{56}, X_{1147}.

Among them, we have the Orthocubic **K006** as already said and also $p\mathcal{K}(X_{69}, X_{315})$, $p\mathcal{K}(X_{219}, X_{3436})$, $p\mathcal{K}(X_{577}, X_{20})$, $p\mathcal{K}(X_{2165}, X_{68})$, $p\mathcal{K}(X_{32} \times X_{2996}, X_{2996})$.

All these cubics have three asymptotes parallel to the Simson lines passing through X_3 and also to the asymptotes of the Orthocubic **K006**.

The most interesting is probably **K690** = $p\mathcal{K}(X_{2165}, X_{68})$ since it contains X_4, X_{68}, X_{485}, X_{486}, X_{637}, X_{638}. See Figure 9.

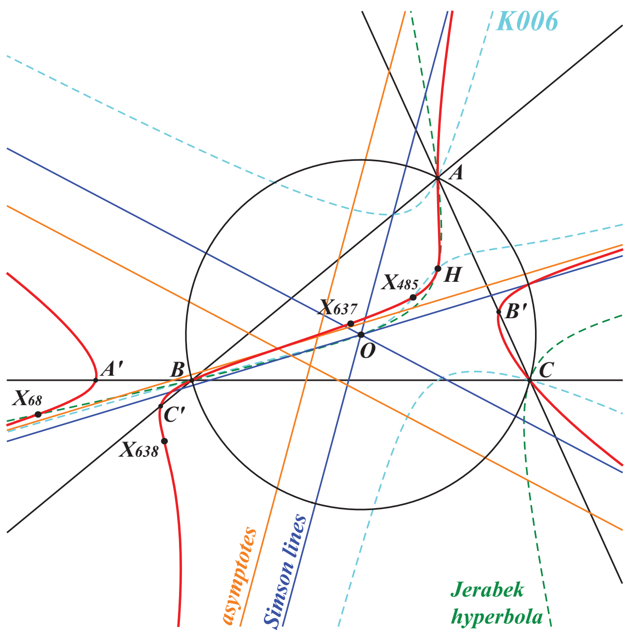

Figure 9. **K690** = $p\mathcal{K}(X_{2165}, X_{68})$

Example 2 : When we consider the cubics having their asymptotes parallel to the Simson lines passing through the incenter $I = X_1$, we find

- $\mathcal{K}(M = X_1, \Omega) = ps\mathcal{K}(X_{31}, X_2, X_6)$ passing through X_6, X_9, X_{19}, X_{478}.

- $\mathcal{K}(M = X_1, P)$ passing through X_4, X_8. It is the anticomplement of $ps\mathcal{K}(X_{31}, X_2, X_1)$ which contains X_1, X_3, X_{56}.

There are two interesting related cubics namely **K691** = $p\mathcal{K}(X_{19}, X_4)$ and **K692** = $p\mathcal{K}(X_6, X_8)$ generating a pencil which contains the decomposed cubic which is the union of the line X_4, X_9 and the circum-conic passing through X_1 and X_4.

Example 3 : The Simson lines passing through the nine point center X_5 form a (decomposed) stelloid and the cubics having their asymptotes parallel to these Simson lines are equilateral cubics.

- $\mathcal{K}(M = X_5, \Omega)$ = **K307** = $ps\mathcal{K}(X_{51}, X_2, X_6)$ passes through X_6, X_{53}, X_{216}, X_{1249}.
- $\mathcal{K}(M = X_5, P)$ passes through X_3, X_4, X_{20}, X_{1670}, X_{1671}. It is the anticomplement of **K026** = $ps\mathcal{K}(X_{51}, X_2, X_3)$ which contains X_3, X_4, X_5.

The most remarkable corresponding cubics are the McCay cubic **K003** = $p\mathcal{K}(X_6, X_3)$, the McCay orthic cubic **K049** = $p\mathcal{K}(X_{53}, X_4)$ and **K096** = $p\mathcal{K}(X_{216}, X_{20})$.

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