Lemoine and Tucker Conics and Circles of a Cubic

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Abstract

We extend the definition and properties of the Tucker conics and Tucker circles of a triangle when this triangle is replaced by a cubic with three real asymptotes.

1 Lemoine, Tucker and Third conics of a triangle

In this first section, we extend the definitions of the well-known Lemoine and Tucker circles of a triangle to obtain the Lemoine and Tucker conics of a triangle.

1.1 Tucker conics, Lemoine conics

Let ABC be a proper triangle and let $L = p : q : r$ be a finite point not lying on its sidelines. An homothety h, with center L and ratio λ, transforms ABC into $A'B'C'$. The line $B'C'$ meets AB at A_c and AC at A_b.

These points are $A_b = (1-\lambda)p : 0 : \lambda p + q + r$ and $A_c = (1-\lambda)p : \lambda p + q + r : 0$.

The points B_a, B_c on $C'A'$ and C_b, C_a on $A'B'$ are defined similarly. See figure 1.

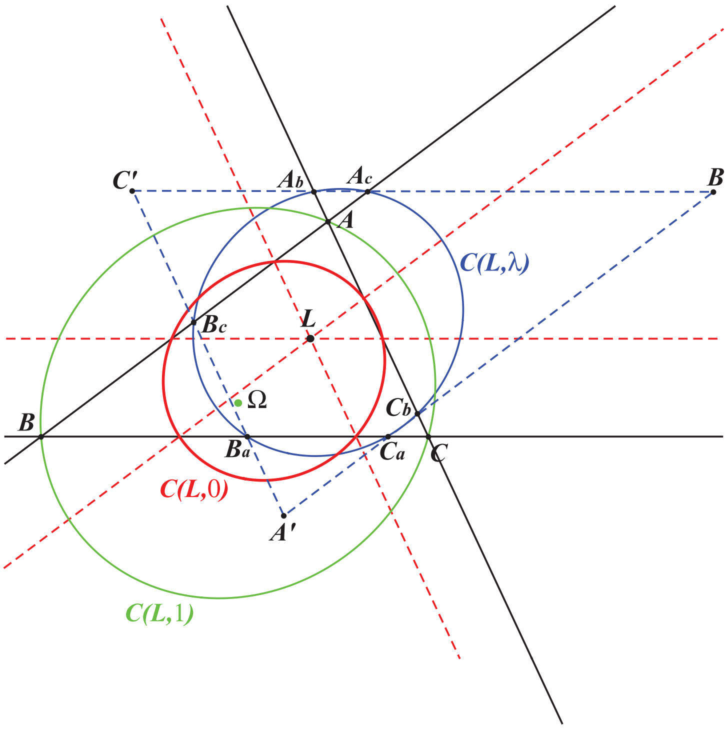

Figure 1: Tucker parallels

By Pascal's theorem, these six points lie on a same conic $\mathcal{C}(L,\lambda)$ with equation

$$\begin{aligned}&\lambda(\lambda-1)\,pqr\,(x+y+z)^2\\&+(\lambda-1)\,(x+y+z)\left(p^2qz+p^2ry+pq^2z+pr^2y+q^2rx+qr^2x\right)\\&+(p+q+r)^2(pyz+qxz+rxy)=0\end{aligned} \tag{1}$$

clearly showing that all these conics, when λ varies, are homothetic since they meet the line at infinity at the same points as the circum-conic with perspector L which is obtained when $\lambda = 1$. This latter conic has equation $pyz + qxz + rxy = 0$ and its center is denoted Ω.

Their centers $T(L, \lambda)$ lie on the line passing through L and the center Ω above and we have :

$$\overrightarrow{LT} = \frac{1+\lambda}{2}\,\overrightarrow{L\Omega}.$$

It follows that the lines B_cC_b, C_aA_c, A_bB_a are parallel to the tangents at A, B, C to the circum-conic above or, equivalently, to the sidelines of the anticevian triangle of L. For this latter reason, these lines will be called antiparallels (with respect to L) to the sidelines of triangle ABC.

The conic $\mathcal{C}(L, \lambda)$ is then called a $L-$Tucker conic of ABC and, in particular, $\mathcal{C}(L, 0)$ and $\mathcal{C}(L, -1)$ are called first and second $L-$Lemoine conics of ABC. Their centers are the midpoint of $L\Omega$ and L respectively.

Remarks :

1. $\mathcal{C}(L, \lambda)$ degenerates if and only if $\lambda(p^2 + q^2 + r^2) + (1 + \lambda^2)(qr + rp + pq) = 0$.

2. In particular, for any point L on the Steiner ellipse, the conic $\mathcal{C}(L, 0)$ is always decomposed into two lines secant at the midpoint of $L\Omega$ and parallel to the asymptotes of the circum-conic $\mathcal{C}(L, 1)$.

3. When L is the centroid G of ABC, $\mathcal{C}(G, \lambda)$ is an ellipse with center G homothetic to the Steiner ellipse.

1.2 Third conics

Each conic $\mathcal{C}(L, \lambda)$ is transformed into another conic $\mathcal{C}'(L, \lambda)$ under the homothety with center $T(L, \lambda)$ and same ratio $\rho = \sqrt{\dfrac{p^2 + q^2 + r^2}{pq + pr + qr} - 1}$, under the condition that L does not lie on Steiner ellipse since ρ would be infinite. See remark below.

These conics $\mathcal{C}'(L, \lambda)$ will be called the $L-$Third conics of ABC, see [8, 9].

Their equations are

$$\begin{aligned}&(pq + pr + qr)(p^2yz + q^2xz + r^2xy - qrx^2 - pry^2 - pqz^2)\\ &+\lambda(x + y + z)(q^2r^2x + p^2r^2y + p^2q^2z - p^2qrx - pq^2ry - pqr^2z) = 0,\end{aligned} \tag{2}$$

hence they form a pencil of homothetic conics (recall they all have the same points at infinity) and they must pass through two other finite points which are the bicentric mates Ω_1 and Ω_2 of the isotomic conjugate L' of L. These points are :

$$L' = \frac{1}{p} : \frac{1}{q} : \frac{1}{r}, \quad \Omega_1 = \frac{1}{q} : \frac{1}{r} : \frac{1}{p}, \quad \Omega_2 = \frac{1}{r} : \frac{1}{p} : \frac{1}{q},$$

in which case the line Δ_L passing through Ω_1 and Ω_2 is

$$q^2r^2x + p^2r^2y + p^2q^2z - p^2qrx - pq^2ry - pqr^2z = 0$$

and the conic $\Gamma_L = \mathcal{C}'(L, 0)$ with equation

$$p^2yz + q^2xz + r^2xy - qrx^2 - pry^2 - pqz^2 = 0$$

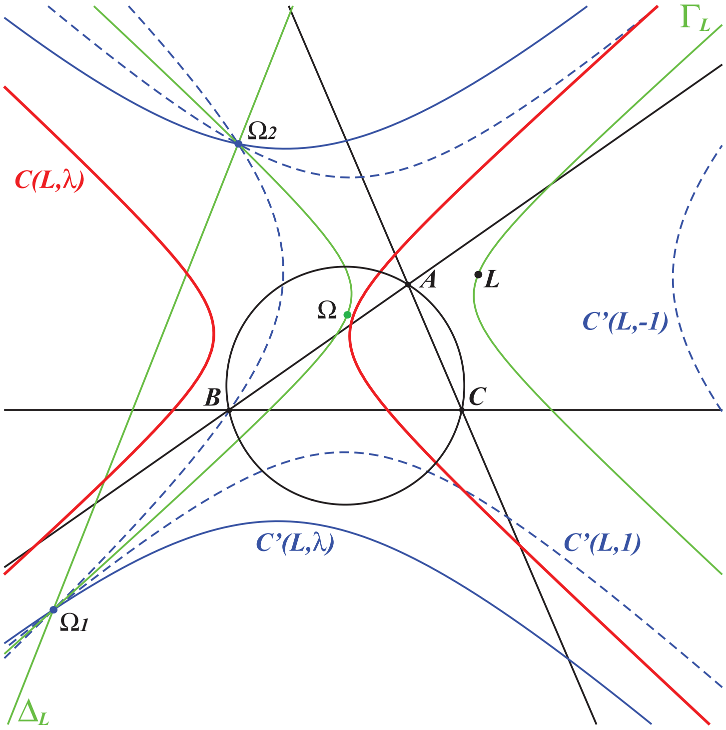

Figure 2: Tucker and Third conics

is that passing through L, Ω, Ω_1, Ω_2 and the two points at infinity of the pencil. See figure 2.

Remark :

When L lies on Steiner ellipse, Δ_L is the line at infinity and the $L-$Third conics are not defined.

1.3 Envelope of the Tucker conics

Let $L \neq G$. When λ varies, the Tucker conics $\mathcal{C}(L, \lambda)$ envelope the inscribed conic $\mathcal{I}(L)$ with perspector L.

Each conic is then bitangent to $\mathcal{I}(L)$ at two points lying on a parallel $\mathcal{P}(L, \lambda)$ to the trilinear polar $\mathcal{P}(L)$ of L. See figure 3.

The equation of $\mathcal{P}(L, \lambda)$ is :

$$(p+q+r)(qrx+rpy+pqz)+2pqr(\lambda-1)(x+y+z)=0,$$

hence $\mathcal{P}(L,1)=\mathcal{P}(L)$.

$\mathcal{P}(L, \lambda)$ meets the line $L\Omega$ at E such that :

$$\begin{aligned}\overrightarrow{LE} =& \frac{2\lambda+1}{4} \times \frac{p^2+q^2+r^2-2qr-2pq-2pr}{p^2+q^2+r^2-qr-pq-pr}\,\overrightarrow{L\Omega} \\ =& \frac{2\lambda+1}{4}\left(1-\frac{pq+pr+qr}{p^2+q^2+r^2-qr-pq-pr}\right)\,\overrightarrow{L\Omega},\end{aligned} \tag{3}$$

hence $\mathcal{P}(L,-1/2)$ is the line that passes through L.

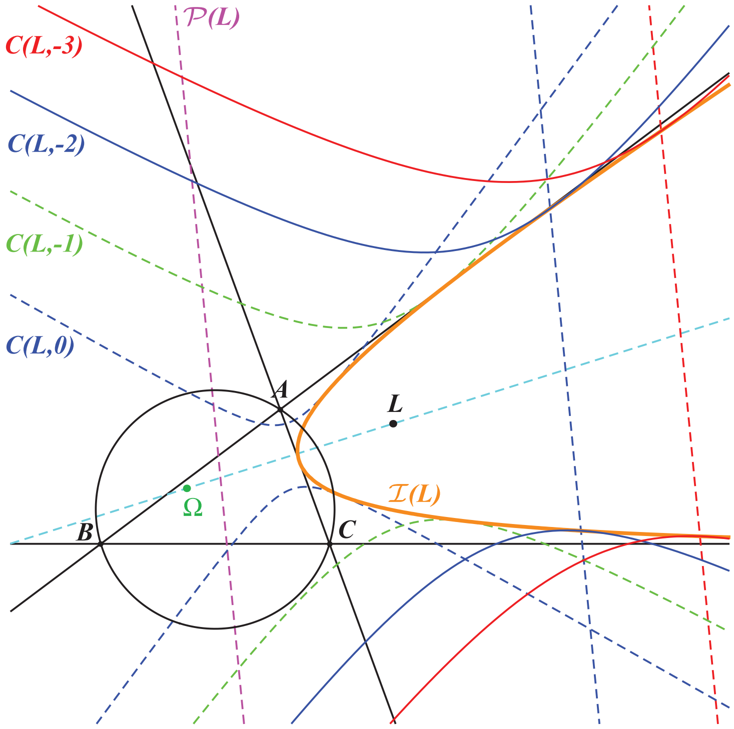

Figure 3: Envelope of the Tucker conics

1.4 Tucker circles and Third circles

It is then clear that these Tucker conics become Tucker circles if and only if L is the Lemoine point X_6 of ABC. See [6] for further properties of these circles. Obviously, Ω_1 and Ω_2 become the Brocard points, Γ_L becomes the Brocard circle of ABC and $\mathcal{I}(L)$ becomes the Brocard ellipse. The corresponding conics $\mathcal{C}'(L, \lambda)$ are called the Third circles in [3].

2 Preliminary properties of a cubic

Since the union of the sidelines of triangle ABC can be seen as a special decomposed cubic curve, we wish to generalize and adapt the results above when the cubic is a proper cubic with three real asymptotes. At first, we recall several properties we shall need in the sequel.

Let $\mathcal{K}$ be a cubic in the plane of the usual reference triangle ABC and let $\mathcal{A}$ be the union of its three asymptotes $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$ which are supposed to be all real in this paper.

These asymptotes meet $\mathcal{K}$ again at three collinear points lying on the satellite line $\mathcal{S}$ of the line at infinity $\mathcal{L}_\infty$. Note that $\mathcal{S}$ is undetermined when the three asymptotes concur at a point lying on the cubic.

$\mathcal{K}$ and $\mathcal{A}$ generate a pencil of cubics $\mathcal{F} = \lambda_1\,\mathcal{K} + \lambda_2\,\mathcal{A}$ which obviously contains the decomposed cubic $\mathcal{L}_\infty^2 \cup \mathcal{S}$ where $\mathcal{L}_\infty^2$ is the line at infinity counted twice.

2.1 Polar conics

Let $P = p : q : r$ be a finite point and let $\mathcal{C}(P, \mathcal{K})$ denote the polar conic of P in $\mathcal{K}$.

Lemma 1. *The polar conics of P in the cubics of $\mathcal{F}$ form a pencil of conics generated by $\mathcal{C}(P, \mathcal{K})$ and $\mathcal{C}(P, \mathcal{A})$.*

Proof. This is obvious since these polar conics can be written

$$\sum_{\text{cyclic}} p\, \frac{\partial \mathcal{F}}{\partial x}(x, y, z) = 0 \iff \lambda_1 \sum_{\text{cyclic}} p\, \frac{\partial \mathcal{K}}{\partial x}(x, y, z) + \lambda_2 \sum_{\text{cyclic}} p\, \frac{\partial \mathcal{A}}{\partial x}(x, y, z) = 0.$$

□

Note that $\mathcal{C}(P, \mathcal{A})$ is a circum-conic of the triangle formed by the asymptotes since it must contain the double points of $\mathcal{A}$. The perspector of $\mathcal{C}(P, \mathcal{A})$ with respect to this triangle is obviously P.

Lemma 2. *The polar conic of P in $\mathcal{L}_\infty^2 \cup \mathcal{S}$ splits into two lines and one of them is $\mathcal{L}_\infty$.*

This again comes from the fact that the polar conic of P must pass through all the double points of $\mathcal{L}_\infty^2 \cup \mathcal{S}$. It follows that all these polar conics have the same points at infinity hence

Lemma 3. *The polar conics of P in the cubics of $\mathcal{F}$ are homothetic.*

Consequently, they must have two other finite points in common P_1, P_2 obviously lying on $\mathcal{C}(P, \mathcal{K})$ and $\mathcal{C}(P, \mathcal{A})$.

Lemma 4. *The points P_1, P_2 lie on the line $\mathcal{L}'$ which is the image of $\mathcal{S}$ under the homothety with center P, ratio $3/2$. See figure 4.*

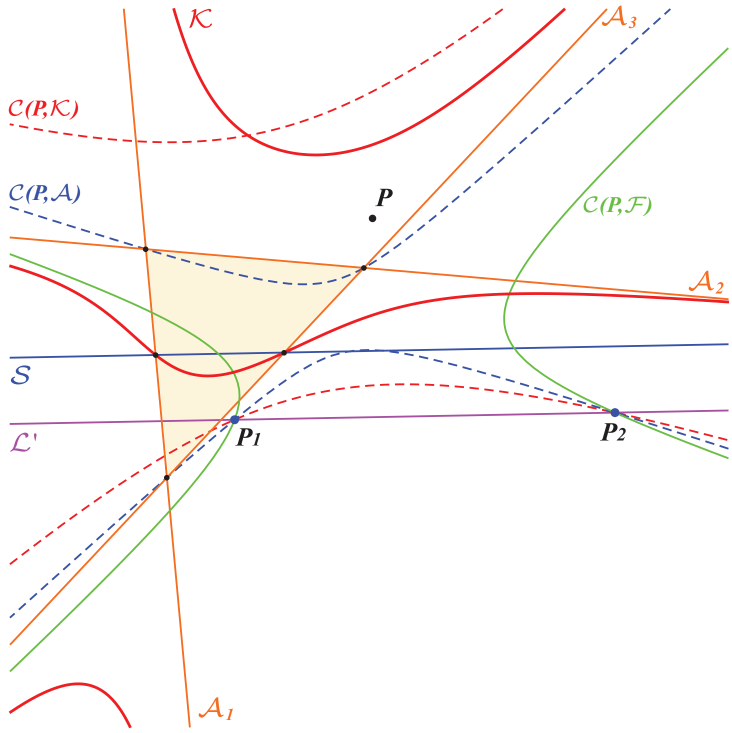

Figure 4: Preliminary properties

Proof. A variable line $\mathcal{L}$ passing through P meets $\mathcal{K}$ at three points M_1, M_2, M_3 and $\mathcal{C}(P,\mathcal{K})$ at two points N_1, N_2. Following Stuyvaert and Cremona (see [7], [1]) we have :

$$\begin{aligned} \frac{3}{PN_1.PN_2} &= \frac{1}{PM_1.PM_2} + \frac{1}{PM_2.PM_3} + \frac{1}{PM_3.PM_1} \\ &= \frac{PM_1 + PM_2 + PM_3}{PM_1.PM_2.PM_3} \end{aligned} \tag{4}$$

and

$$\frac{1}{PN_1} + \frac{1}{PN_2} = \frac{2}{3}\left(\frac{1}{PM_1} + \frac{1}{PM_2} + \frac{1}{PM_3}\right), \tag{5}$$

where all the distances are considered to be algebraic.

If $\mathcal{K}$ is $\mathcal{L}_\infty^2 \cup \mathcal{S}$ then $M_2 = M_3 = N_2 = \infty$ hence (5) gives

$$\frac{1}{PN_1} = \frac{2}{3}\,\frac{1}{PM_1} \iff PN_1 = \frac{3}{2}\,PM_1,$$

and, since M_1 lies on $\mathcal{S}$, N_1 must lie on the line $\mathcal{L}'$. □

2.2 Lemoine point of a cubic

Generally speaking, there is one and only one point K whose polar conic is a circle but, for some special cubics, one can find a pencil of circular polar conics which we do not consider in this paper. See [7] for further details.

When K is uniquely defined, for any line $\mathcal{L}$ passing through K the product $KN_1.KN_2$ is constant : this is the power $\mathcal{P}$ of K with respect to its (circular) polar conic.

It follows that the quantities $PM_1+PM_2+PM_3$ and $PM_1.PM_2.PM_3$ are proportional with a common fixed ratio for any line $\mathcal{L}$.

When the cubic $\mathcal{K}$ is the union of the sidelines of ABC, K is the Lemoine point X_6 of ABC and its polar conic $\mathcal{C}$ is the circumcircle of ABC. For this reason, K will be called the Lemoine point of the general cubic $\mathcal{K}$ assuming that it is defined. In such case, K is also the Lemoine point of the triangle $\mathcal{T}$ formed by the asymptotes of $\mathcal{K}$ and then, its polar conic in the cubic formed by the sidelines of $\mathcal{T}$ is the circumcircle of $\mathcal{T}$.

3 Cubic with non-concurring asymptotes

Let us now consider a cubic $\mathcal{K}$ with three non-concurring asymptotes $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$ and Lemoine point K. In this case, K is the Lemoine point of the triangle $\mathcal{T}$ formed by the asymptotes and the polar conic of K with respect to the cubic $\mathcal{A}$ which is the union of the asymptotes is the circumcircle of this same triangle $\mathcal{T}$.

The properties below are illustrated with the cubic K149 $= n\mathcal{K}_0(X_2, X_6)$.

3.1 Tucker conics of a cubic

Let L be a finite point not lying on one asymptote of $\mathcal{K}$ and let λ be a real number. As above, the asymptotes $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$ are transformed into three lines $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_3$ under the homothety h.

$\mathcal{K}$ and the union $\mathcal{L}$ of these three lines generate a pencil of cubics having the same points at infinity and intersecting again at six other points that must lie on a same conic $\mathcal{C}(L, \lambda)$ we call a Tucker conic of $\mathcal{K}$ associated with L.

A rather difficult computation gives the equation of $\mathcal{C}(L, \lambda)$ namely :

$$\begin{aligned}(\lambda-1)^3(x+y+z)^2\ \Phi_3+(\lambda-1)^2(p+q+r)(x+y+z)\ \Phi_2 \\ +(\lambda-1)(p+q+r)^2\ \Phi_1+(p+q+r)^3(x+y+z)\ \Phi_0=0\end{aligned} \tag{6}$$

where :

- Φ_3 is a constant, depending of L and $\mathcal{K}$, which would be zero if L be on the asymptotes of $\mathcal{K}$,
- $\Phi_2 = 0$ is the polar line of L in $\mathcal{A}$,
- $\Phi_1 = 0$ is the polar conic of L in $\mathcal{A}$,
- $\Phi_0 = 0$ is the satellite line $\mathcal{S}$ of $\mathcal{L}_\infty$ in $\mathcal{K}$.

Note that Φ_1 rewrites under the form $\Phi_1 = (x+y+z)\Psi_1 + k\Psi_2$ where :

- $\Psi_1 = 0$ is the line $\mathcal{L}'$, image of $\mathcal{S}$ under the homothety with center L, ratio $3/2$,
- $\Psi_2 = 0$ is the polar conic of L in $\mathcal{K}$,
- k is a constant, depending of L and $\mathcal{K}$, which is zero if and only if $\mathcal{K}$ has a node at infinity. In this latter case, $\mathcal{C}(L, \lambda)$ splits into $\mathcal{L}_\infty$ and another line.

When λ varies, all these conics $\mathcal{C}(L, \lambda)$ share the same points at infinity and these points are those of the polar conics of L in $\mathcal{K}$ and in $\mathcal{A}$, this latter being the circum-conic with perspector L of the triangle bounded by the asymptotes.

Let $M = u : v : w$ be a point and m its polar line in $\mathcal{C}(L, \lambda)$. The polar line of M in the first term $(\lambda-1)^3(x+y+z)^2\ \Phi_3$ of (6) is already $\mathcal{L}_\infty$ hence m is $\mathcal{L}_\infty$ if and only if the polar line of M in the rest of the equation is $\mathcal{L}_\infty$. This gives a condition of degree 2 in λ showing that, when λ varies, the locus of M such that $m = \mathcal{L}_\infty$ is generally a conic $\mathcal{H}(L)$, the locus of the center $\Omega(L, \lambda)$ of $\mathcal{C}(L, \lambda)$.

More precisely, $\mathcal{H}(L)$ is a proper hyperbola if and only if the polar conic of L in $\mathcal{A}$ is not a parabola and the polar line of L in $\mathcal{A}$ and the satellite line $\mathcal{S}$ are not parallel.

In this case, the asymptotes of $\mathcal{H}(L)$ are the conjugated diameters of these two latter lines in the polar conic of L in $\mathcal{A}$ whose center is that of $\mathcal{H}(L)$.

When the two conditions above are not fulfilled, $\mathcal{H}(L)$ becomes a double line analogous to the Brocard axis of the configuration we had studied at the beginning of this paper.

The special cases $\mathcal{C}(L, 0)$, $\mathcal{C}(L, -1)$ are called first and second Lemoine conics of $\mathcal{K}$ relatively to L.

The following figure 5 represents K149 with two Tucker conics associated with $L = O = X_3$, the circumcenter of ABC. Note that $\mathcal{C}(X_3, 0)$ is a first Lemoine conic.

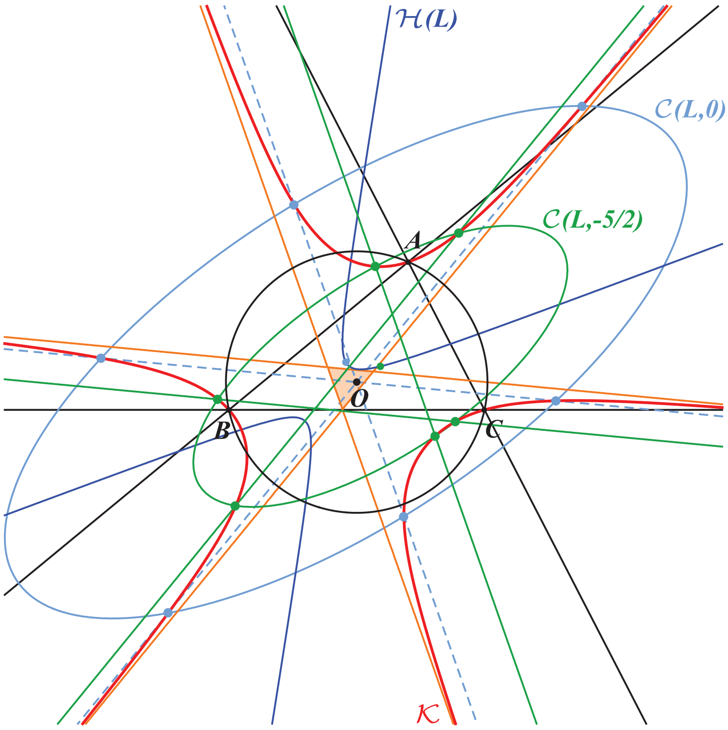

Figure 5: The cubic K149 with two Tucker conics

3.2 First Lemoine circle of a cubic

This is $\mathcal{C}(K, 0)$ where K is the Lemoine point of $\mathcal{K}$.

Nevertheless, in this case, a more direct and simpler approach can be used. Indeed, let $\mathcal{L}$ be the line parallel at K to one of the asymptotes $\mathcal{A}$.

With the same notations as above, $\mathcal{L}$ meets $\mathcal{K}$ at M_1, M_2, $M_3 = \infty$ and $\mathcal{C}$ at two points N_1, N_2. Since $\dfrac{1}{KM_3} = 0$, equation (4) above gives

$$\frac{3}{KN_1.KN_2} = \frac{1}{KM_1.KM_2} \iff KM_1.KM_2 = \frac{P}{3},$$

where P is the power of K with respect to $\mathcal{C}$.

Similarly, with the two other asymptotes, we finally obtain six points on the cubic such that

$$KM_1.KM_2 = KM_1'.KM_2' = KM_1''.KM_2'' = \frac{P}{3}$$

It follows that these six points must lie on a same circle we shall call the first Lemoine circle of $\mathcal{K}$. See figure 6.

3.3 Second Lemoine circle of a cubic

$\mathcal{T}'$ is the reflection of $\mathcal{T}$ about K. Since this reflection is the homothety whose ratio is -1, its sidelines meet the cubic again at six points lying on a same circle $\mathcal{C}(K, -1)$ we shall call the second Lemoine circle of $\mathcal{K}$. See figure 6.

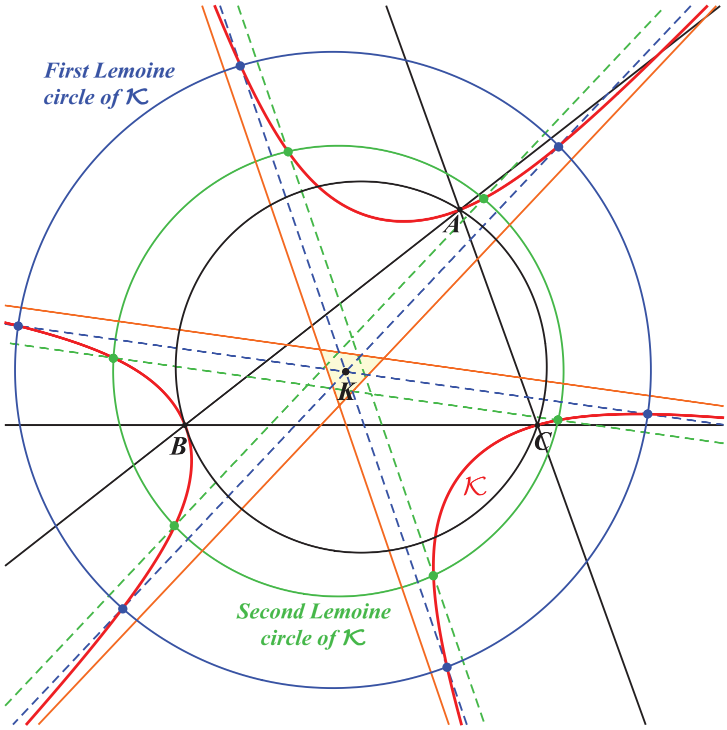

Figure 6: The cubic K149 with its two Lemoine circles

3.4 Tucker circles of a cubic

We now transform these asymptotes $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$ under the homothety h with center K, ratio λ, to obtain three lines $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_3$ obviously parallel to the asymptotes.

The polar conic of K being the circumcircle of the triangle formed by the asymptotes, these lines must meet the cubic again at six points lying on a same circle we shall call a Tucker circle $\mathcal{C}(K, \lambda)$ of $\mathcal{K}$. Naturally, with $\lambda = 0$, $\lambda = -1$, we have the two Lemoine circles. The figure 7 shows the case $\lambda = -5/2$.

3.5 Centers of the Tucker circles of a cubic

We have seen that the locus of the centers of the Tucker conics of a cubic is in general a proper hyperbola and, naturally, this remains true for the Tucker circles but now, it is possible to be more explicit.

After some computations, we find that the locus of the center $\Omega(K, \lambda)$ of $\mathcal{C}(K, \lambda)$ is the hyperbola $\mathcal{H}(K)$ with the following properties :

- its center is the center Ω' of the circle which is the polar conic of K in $\mathcal{A}$, the circum-circle of the triangle formed by the asymptotes,
- one of its asymptotes is the line $K\Omega'$, the Brocard axis of this latter triangle,
- the other asymptote is the line $\Omega\Omega'$ where Ω is the center of the polar conic of K in $\mathcal{K}$,
- it contains the midpoint of ΩK, the center of the first Lemoine circle of $\mathcal{K}$.

When these three points K, Ω, Ω' are collinear the hyperbola degenerates into a double line, the Brocard axis above, which is the “classical” situation.

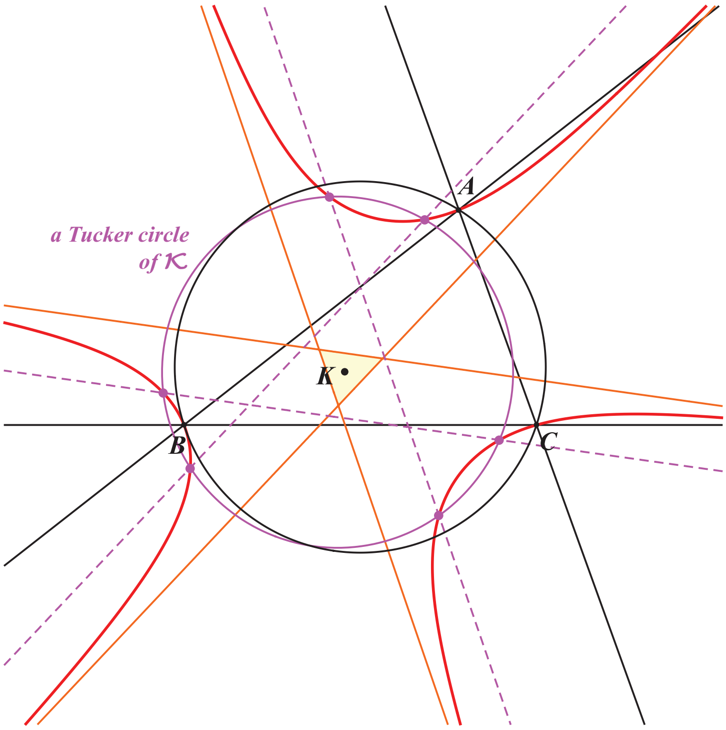

Figure 7: The cubic K149 with one of its Tucker circles

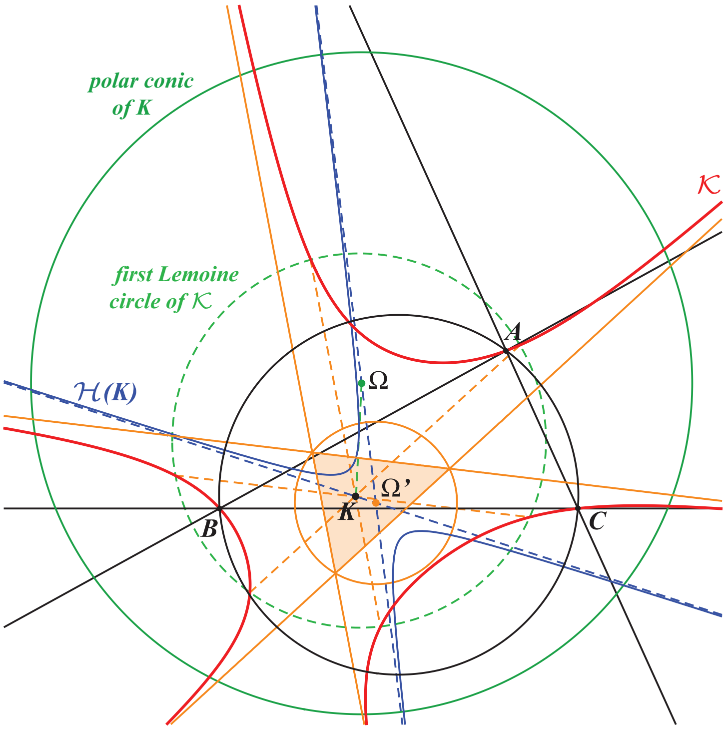

Figure 8: The cubic K149 with the hyperbola $\mathcal{H}(K)$

4 Cubic with concurring asymptotes

When the cubic $\mathcal{K}$ has three concurring asymptotes, its Lemoine point K is the point of concurrence since its polar conic splits into the line at infinity $\mathcal{L}_\infty$ and another (finite) line $\mathcal{L}'$. Let us take the line $\mathcal{L}$ as one of the asymptotes so that $KM_2 = KM_3 = KN_2 = \infty$. From (5) we obtain

$$\frac{1}{KN_1} = \frac{2}{3}\,\frac{1}{KM_1} \iff KM_1 = \frac{2}{3}\,KN_1,$$

which shows that $\mathcal{L}'$ is the image of the satellite line $\mathcal{S}$ of $\mathcal{L}_\infty$ under the homothety with center K and ratio $\frac{3}{2}$.

When the asymptotes concur on the cubic, $\mathcal{L}'$ is the tangent at this point to the cubic and this point must be a point of inflection.

4.1 A special case : the cubic K082

We illustrate and develop the properties above with the cubic K082 $= n\mathcal{K}_0(X_6, X_2)$ since it presents a configuration very similar to that of the classical literature. Indeed, its Lemoine point is actually the Lemoine point X_6 of ABC and the asymptotes are the parallels at X_6 to the sidelines of ABC meeting these sidelines at six points lying on the first Lemoine circle or triplicate ratio circle of ABC. Note that $\mathcal{L}_2$ and $\mathcal{S}$ are the trilinear polars of X_6 and X_{251} respectively.

K082 is the only circum-cubic having three concurring asymptotes parallel at X_6 to the sidelines of ABC[1]. See figure 9.

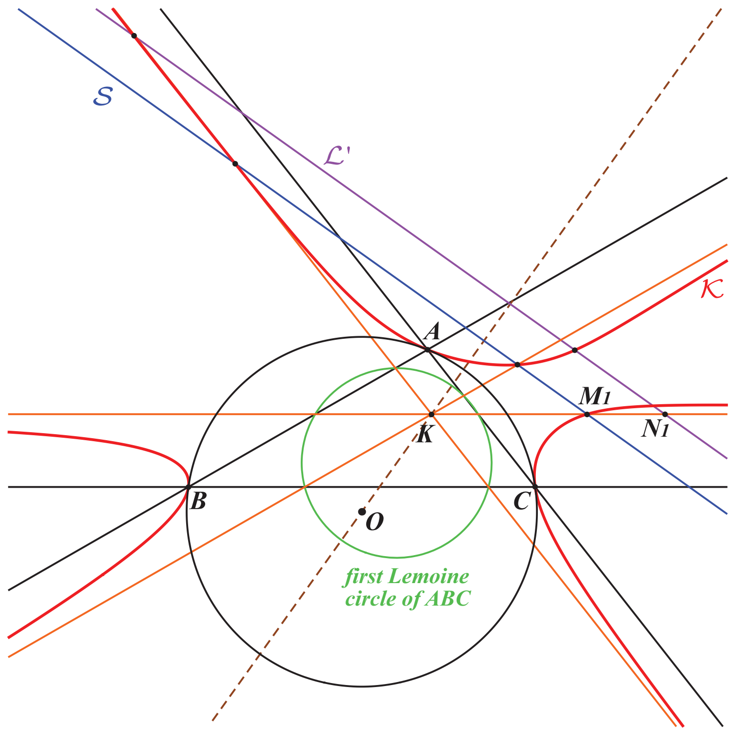

Figure 9: The cubic K082 $= n\mathcal{K}_0(X_6, X_2)$

Let us now consider the triangle $A'B'C'$ homothetic of ABC under the homothety with center X_6 and ratio λ. It is known that the sidelines of $A'B'C'$ meet the parallels at X_6 to the sidelines of ABC at six points lying on a same circle called Tucker circle (of ABC) hereby denoted (T_λ).

[1]More generally, $n\mathcal{K}_0(X, X_2)$ is the only proper circum-cubic having three real asymptotes parallel to the sidelines of ABC and concurring at X.

These same sidelines of $A'B'C'$ meet the cubic at three points on $\mathcal{L}_\infty$ and six other points that also lie on a same circle (C_λ) we shall call Tucker circle of the cubic.

A computation gives the equation of (C_λ) :

$$\begin{aligned}(\lambda-1)^2\,(x+y+z)\sum_{\text{cyclic}} b^2c^2(b^2+c^2+a^2\lambda)x \\ +\lambda\,(a^2+b^2+c^2)^2(a^2yz+b^2zx+c^2xy)=0\end{aligned} \tag{7}$$

to be compared with the equation of (T_λ) which is :

$$\begin{aligned}(\lambda-1)\,(x+y+z)\sum_{\text{cyclic}} b^2c^2(b^2+c^2+a^2\lambda)x \\ +(a^2+b^2+c^2)^2(a^2yz+b^2zx+c^2xy)=0\end{aligned} \tag{8}$$

clearly showing that these two circles belong to a pencil containing the circumcircle of ABC which is obtained when $\lambda=1$. This is the only case where the two circles coincide and also the only case for which (C_λ) is a Tucker circle for ABC.

The centers T_λ, C_λ of (T_λ), (C_λ) both lie on the Brocard axis and are given by :

$$\overrightarrow{KT_\lambda}=\frac{1+\lambda}{2}\,\overrightarrow{KO} \quad \text{and} \quad \overrightarrow{KC_\lambda}=\frac{1+\lambda^2}{2\lambda}\,\overrightarrow{KO}.$$

The radical axis of (T_λ), (C_λ) is the perpendicular to the Brocard axis at point A_λ such that :

$$\overrightarrow{KA_\lambda}=\frac{2+\lambda}{3-\cot^2\omega}\,\overrightarrow{KO}$$

where ω is the Brocard angle. See figure 10.

Since (T_0), (T_{-1}) are the first and second Lemoine circles of ABC, we shall say that (C_0), (C_{-1}) are the first and second Lemoine circles of the cubic. Recall that (T_1), (C_1) are both the circumcircle of ABC.

With $\lambda=-1-2\cos A\cos B\cos C$, we find the Taylor circle but its counterpart in the cubic does not usually intersect the sidelines of triangle $A'B'C'$ which gives it very little interest.

4.2 The general case

We now intend to generalize the properties above for a circum-cubic having three asymptotes concurring at a certain point K but not necessarily parallel to the sidelines of ABC.

We need construct a proper triangle $\mathcal{T}_K$ whose sidelines are parallel to these three asymptotes and whose Lemoine point is K.

Let $\mathcal{L}_i, i\in\{1,2,3\}$ be the line passing through one point at infinity of the cubic and the complement of the isotomic conjugate of this point, obviously on the Steiner inscribed ellipse.

These three lines bound a triangle whose Lemoine point is K' and it is enough to transform these lines under the translation that carries K' onto K to obtain our requested triangle $\mathcal{T}_K$.

Applying this method to the cubic K082 clearly shows that $\mathcal{T}_K$ is then ABC itself.

The following figure 11 shows $n\mathcal{K}_0(X_4,X_4)$, a cubic with three asymptotes concurring at the centroid X_2 of ABC hence Lemoine point K of the cubic.

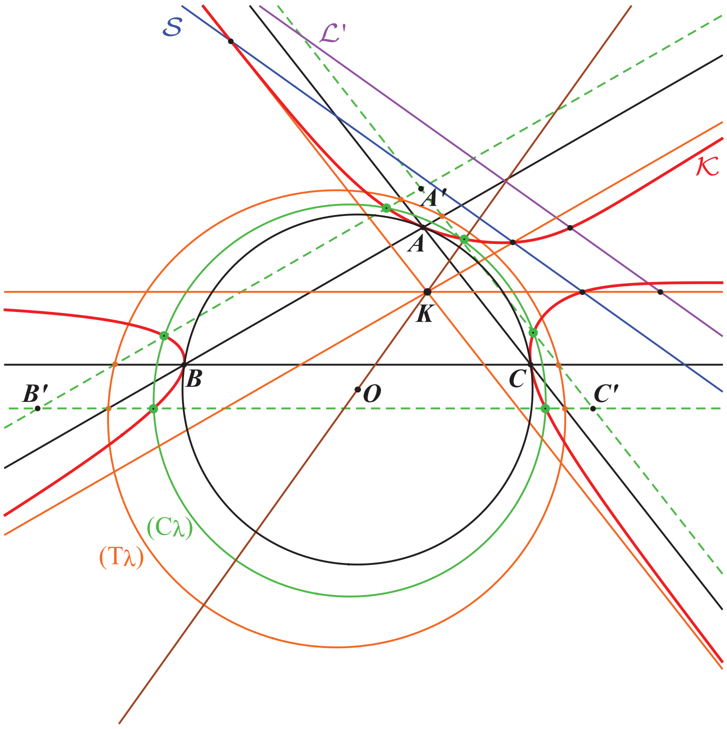

Figure 10: Tucker circles in the cubic K082

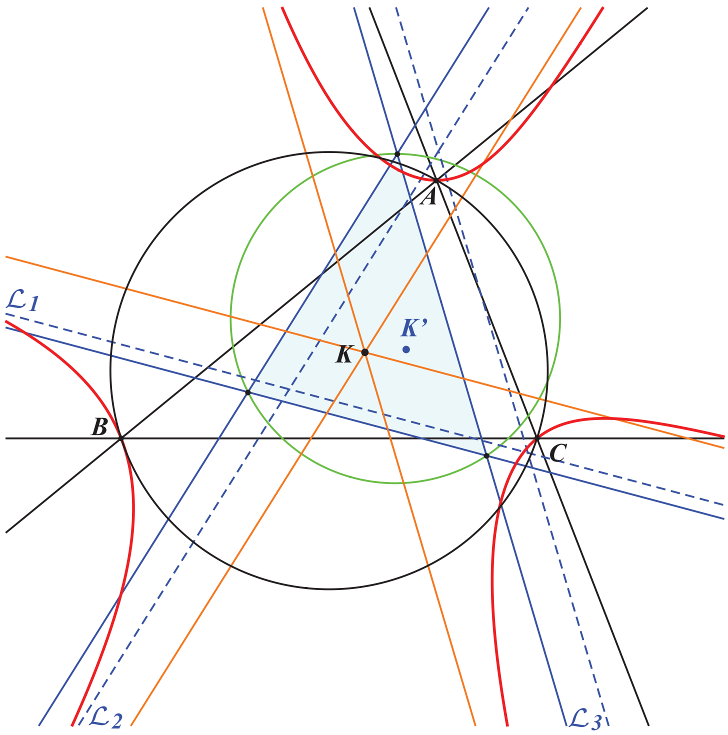

Figure 11: $n\mathcal{K}_0(X_4, X_4)$ and triangle $\mathcal{T}_K$

This triangle $\mathcal{T}_K$ is transformed under the homothety h with center K and ratio λ to give another triangle $\mathcal{T}_{K,\lambda}$ whose sidelines meet the cubic at six finite points. After some computations we find that these six points lie on a same circle which is a Tucker circle of the cubic.

When $\lambda = 0$, this circle splits into $\mathcal{L}_\infty$ and its satellite line with respect to the cubic since $\mathcal{T}_{K,0}$ is the union of the asymptotes of the cubic.

The following figure 12 shows $n\mathcal{K}_0(X_4, X_4)$ again with two of these Tucker circles. It appears that the locus of the centers of these circles is no longer a line but here a hyperbola passing through X_{30} and another infinite point unlisted in ETC.

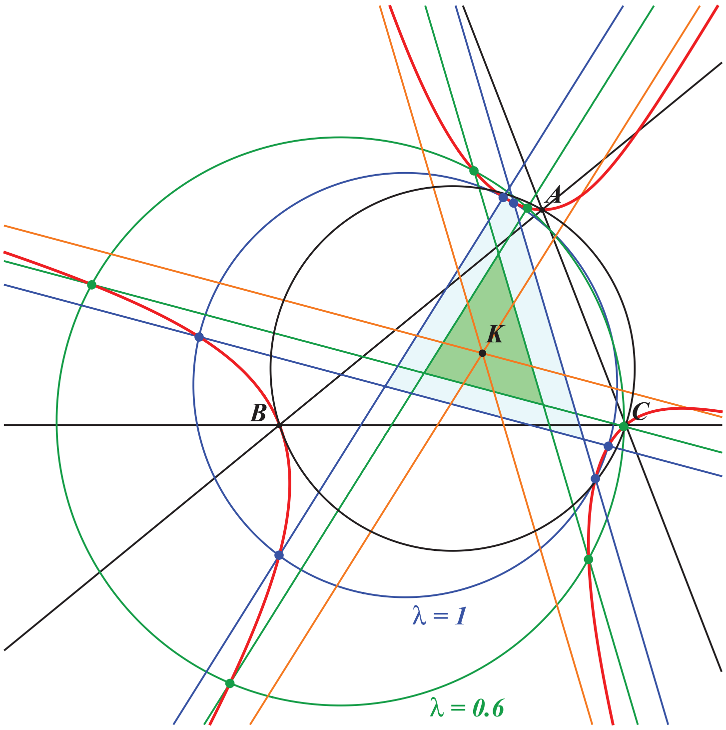

Figure 12: $n\mathcal{K}_0(X_4, X_4)$ and two Tucker circles

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