Cubics related with Concurrent Tangents

Part 1 : Circles

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Abstract

We study several families of cubics related with concurrent tangents. We shall meet many documented cubics and we shall see how the Napoleon cubic **K005** and the second Brocard cubic **K018** are particularly involved.

Notations : if P is a point in the plane of the reference triangle ABC, we denote by :

- $\mathbf{g}P$, the isogonal conjugate of P,
- $\mathbf{t}P$, the isotomic conjugate of P,
- $\mathbf{c}P$, the complement of P,
- $\mathbf{a}P$, the anticomplement of P,
- $\mathbf{i}P$, the inverse of P in the circumcircle $(\mathcal{O})$ of ABC,
- $\mathbf{gig}P$, the antigonal of P,
- $\mathbf{o}P$, the orthocorrespondent of P, see [4].

1 Theorems and definitions

1.1 The cubics $\mathcal{K}(P)$, $\mathcal{K}'(P)$ and $\mathcal{K}''(P)$

Let $P = p : q : r$ be a finite point not lying on the sidelines of ABC and let M be a variable point. Let $\mathcal{T}_A$, $\mathcal{T}_B$, $\mathcal{T}_C$ be the tangents at A, B, C to the circles passing through P, M and A, B, C respectively.

Theorem 1 (Definition of $\mathcal{K}(P)$). *The tangents $\mathcal{T}_A$, $\mathcal{T}_B$, $\mathcal{T}_C$ are concurrent (at N) if and only if M lies on a circular circumcubic $\mathcal{K}(P)$ passing through P, $\mathbf{g}P$, $\mathbf{i}P$, the antigonal $\mathbf{gig}P$ of P and several other points described below.*

Theorem 2 (Definition of $\mathcal{K}'(P)$). *The locus of the common point N of these three tangents is another circular circumcubic $\mathcal{K}'(P)$ passing through the circumcenter O, P, $\mathbf{ga}P$, $\mathbf{ig}P$, $\mathbf{igag}P$ and several other points described below.*

Theorem 3 (Definition of $\mathcal{K}''(P)$). *$\mathcal{K}(P)$ and $\mathcal{K}'(P)$ generate a pencil of circular circumcubics passing through P. This pencil contains one and only one cubic without term in xyz denoted by $\mathcal{K}''(P)$. $\mathcal{K}''(P)$ contains the isodynamic points X_{15}, X_{16} and several other points described below.*

These cubics $\mathcal{K}''(P)$ form a net which can be generated by three decomposed cubics. One of them is the union of the sideline BC and the Apollonian circle passing through A, the other cubics being defined similarly.

In the sequel, we shall denote by $[P,Q,R,S]$, $[P,Q,R,S]'$, $[P,Q,R,S]''$ the coresidual of the points P, Q, R, S in the cubics $\mathcal{K}(P)$, $\mathcal{K}'(P)$, $\mathcal{K}''(P)$ respectively. Recall that any conic passing through four points of a cubic meets the cubic again at two other points collinear with the coresidual of the four points.

1.2 Construction of these cubics

Let E be the second common point of the Euler line and the rectangular circum-hyperbola passing through P (and $\mathbf{gig}P$) . Let $\mathcal{H}$ be the rectangular hyperbola passing through O, P, E having the same asymptotic directions as the previous hyperbola. Let E_1 be the antipode of P on $\mathcal{H}$, a point on the line E, $\mathbf{gig}P$.

A variable line (δ) through P meets $\mathcal{H}$ again at E_2. Let (γ) be the isogonal transform of the anticomplement of (δ). The line E_1E_2 meets the conic (γ) at two points N_1, N_2 on $\mathcal{K}'(P)$.

The perpendicular at A to the line AN_i meets the perpendicular bisector of AP at a point center of a circle passing through P and meeting (δ) again at M_i on $\mathcal{K}(P)$.

Note that the lines N_1, N_2 and $\mathbf{g}M_1, \mathbf{g}M_2$ are parallel. The line N_1, N_2 contains the fixed point E_1 on $\mathcal{K}'(P)$ and the line $\mathbf{g}M_1, \mathbf{g}M_2$ contains a fixed point $\mathbf{g}P_8$ whose isogonal conjugate P_8 is a point on $\mathcal{K}(P)$. It follows that the circumconic through M_1 and M_2 must contain P_8.

At last, the construction of $\mathcal{K}''(P)$ derives from that of orthopivotal cubics. See below.

2 The cubics $\mathcal{K}(P)$

2.1 Other properties of $\mathcal{K}(P)$

Proposition 1. *$\mathcal{K}(P)$ contains Q if and only if $\mathcal{K}(Q)$ contains P.*

Proposition 2. *The isogonal transform of $\mathcal{K}(P)$ is $\mathcal{K}(\mathbf{g}P)$. The corresponding singular foci are inverse in the circumcircle.*

These two cubics generate a pencil of circular circumcubics passing through P and $\mathbf{g}P$ that contains the isogonal circular pivotal cubic, here denoted $p\mathcal{K}(P)$, with pivot the infinite point of the line passing through P and $\mathbf{g}P$. It follows that $\mathcal{K}(P)$ and $\mathcal{K}(\mathbf{g}P)$ meet at two other isogonal conjugate points lying on a parallel to the line through P and $\mathbf{g}P$ (the real asymptote of $p\mathcal{K}(P)$). This latter parallel passes through the second intersection of the diagonal rectangular hyperbola passing through the in/excenters and the midpoint M of P and $\mathbf{g}P$ with the line passing through G and M. Note that this hyperbola is the polar conic of the real infinite point of the line $P\mathbf{g}P$ in $p\mathcal{K}(P)$.

This pencil also contains an isogonal non-pivotal focal cubic, here denoted $n\mathcal{K}_0(P)$, the locus of inscribed conics with center on the line passing through G and the midpoint of P and $\mathbf{g}P$. This line is called the axis (or the orthic line[1]) of $n\mathcal{K}_0(P)$. Hence, this focal cubic contains the four foci of the inscribed Steiner ellipse. See figure 1.

2.2 Other points on the cubic

Recall that $\mathcal{K}(P)$ passes through P, $\mathbf{g}P$, $\mathbf{i}P$, $\mathbf{gig}P$.

- The real infinite point ∞P of $\mathcal{K}(P)$ is that of the line passing through P and $\mathbf{cg}P$.

[1]The orthic line of a circular cubic is the locus of points whose polar conic is a rectangular hyperbola. It is parallel to the real asymptote.

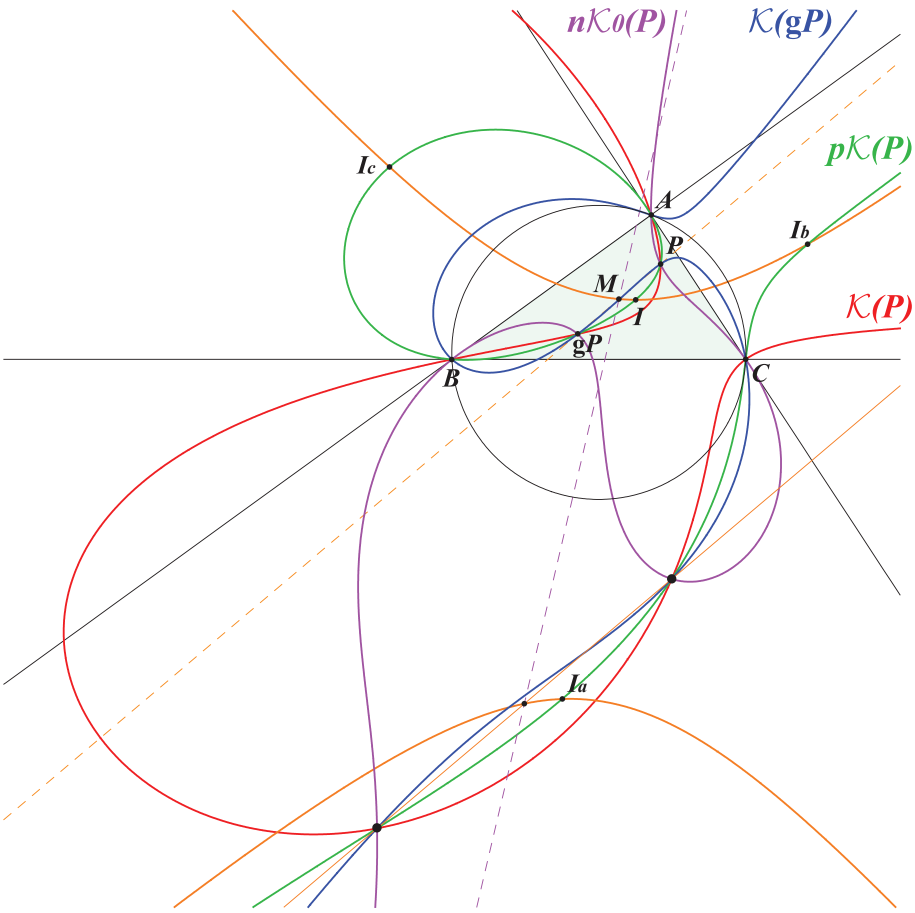

Figure 1: The pencil $\mathcal{K}(P)$, $\mathcal{K}(\mathbf{g}P)$

- The sixth common point P_1 of $\mathcal{K}(P)$ and $(\mathcal{O})$ is the isogonal conjugate of the infinite point of the line passing through P and $\mathbf{ag}P$.

- The third point P_2 on the line P, P_1 also lies on the line $\mathbf{g}P, \mathbf{gig}P$.

- The third point P_3 on the line $\mathbf{g}P, \mathbf{i}P$ also lies on the line $P, \infty P$. P_3 lies on the two circumconics passing through P, P_1 and $\mathbf{g}P, \mathbf{gig}P$. Hence $[A, B, C, P_3] = P_2$.

- The third point P_4 on the line $P, \mathbf{i}P$ also lies on the rectangular circumhyperbola passing through P and $\mathbf{gig}P$. Hence $[A, B, C, \mathbf{gig}P] = \mathbf{i}P$.

- The circumconic through P and $\mathbf{g}P$ meets $\mathcal{K}(P)$ at a sixth point P_5.

- P_6, the third point of $\mathcal{K}(P)$ on the line $\mathbf{g}P, P_4$. P_6 is the orthoassociate of $\mathbf{g}P$ i.e. its inverse in the polar circle.

- P_7, the third point of $\mathcal{K}(P)$ on the line $P, \mathbf{g}P$.

- P_8, the third point of $\mathcal{K}(P)$ on the line $\mathbf{g}P, \infty P$. P_8 also lies on the circumconic through P_1 and $\mathbf{g}P$. Hence $[A, B, C, P_1] = \infty P$.

- P_9, the third point of $\mathcal{K}(P)$ on the line $P, \mathbf{gig}P$, the second common point of the circles $\mathbf{i}P, P, P_1$ and $\mathbf{i}P, \mathbf{g}P, \mathbf{gig}P$.

- $P_{10} = P_1, P_3 \cap P_4, \mathbf{gig}P \cap \mathbf{g}P, P_5$. Hence $P_{10} = [A, B, C, P]$.
- $P_{11} = P, P_5 \cap P_3, \mathbf{gig}P \cap P_1, P_8$. Hence $P_{11} = [A, B, C, \mathbf{g}P]$.
- P_{12}, the third point of $\mathcal{K}(P)$ on the line $P_1, \mathbf{g}P$.
- $P_{13} = P_1, P_6 \cap \mathbf{i}P, \mathbf{gig}P$.
- P_{14}, the third point of $\mathcal{K}(P)$ on the line $\mathbf{i}P, \infty P$. P_{14} lies on the two circumconics passing through $\mathbf{i}P, P_1$ and $\infty P, \mathbf{gig}P$
- $P_{15} = P_1, \mathbf{i}P \cap \infty P, \mathbf{gig}P$. Hence $P_{15} = [A, B, C, P_{14}]$.
- P_{16}, the third point of $\mathcal{K}(P)$ on the line P, P_6.
- $P_{17} = P, P_8 \cap P_6, P_{14}$.
- $P_{18} = P, P_{12} \cap P_6, P_{15} \cap P_8, P_{10}$.

2.3 Special cubics $\mathcal{K}(P)$

Proposition 3. *$\mathcal{K}(P)$ is a focal cubic if and only if P lies on the Napoleon cubic* **K005**.

In this case, the orthic line of $\mathcal{K}(P)$ contains P (and $\mathbf{cg}P$). Hence, the singular focus of $\mathcal{K}(P)$ is the reflection of the center of the polar conic of ∞P about this orthic line.

See table 1 for examples of these cubics.

Proposition 4. *$\mathcal{K}(P)$ is a pivotal cubic if and only if P lies on the second Brocard cubic* **K018**. *In this case, its pole lies on* **K222**, *its pivot lies on the Kiepert hyperbola, its isopivot lies on the Brocard axis. Furthermore, two points P on* **K018** *collinear with G give the same cubic.*

Recall that **K222** is the conico-pivotal cubic with singularity K and root X_{512}. It contains X_6, X_{111}, X_{187}, X_{232}, X_{248}, X_{385}.

See table 2 for examples of these cubics.

3 The cubics $\mathcal{K}'(P)$

3.1 Other properties of $\mathcal{K}'(P)$

Proposition 5. *$\mathcal{K}'(P)$ contains Q if and only if P lies on the pivotal isocubic with pivot $\mathbf{gi}P$ and isopivot Q i.e. the cubic $p\mathcal{K}(Q \times \mathbf{gi}Q, \mathbf{gi}Q)$.*

These are the circular pivotal cubics we meet in **Table 12** of [3].

Proposition 6. *The inverse of $\mathcal{K}'(P)$ in the circumcircle is $\mathcal{K}'(\mathbf{g}P)$.*

$\mathcal{K}'(P)$ and $\mathcal{K}'(\mathbf{g}P)$ generate a pencil of circular circumcubics passing through O and meeting the circumcircle again at the same point Q_1 (see below).

This pencil contains the degenerate cubic which is the union of $(\mathcal{O})$ and the line passing through O and the crosspoint of P and $\mathbf{g}P$, in other words the pole of the line $P\mathbf{g}P$ in the circumconic passing through P and $\mathbf{g}P$.

It also contains a pivotal cubic, here denoted $p\mathcal{K}'(P)$, with pivot Q_1 and isopivot O. This latter cubic is self-inverse in $(\mathcal{O})$.

All these cubics must contain two other (not always real) points, inverse in $(\mathcal{O})$, lying on the line passing through O above. See figure 2.

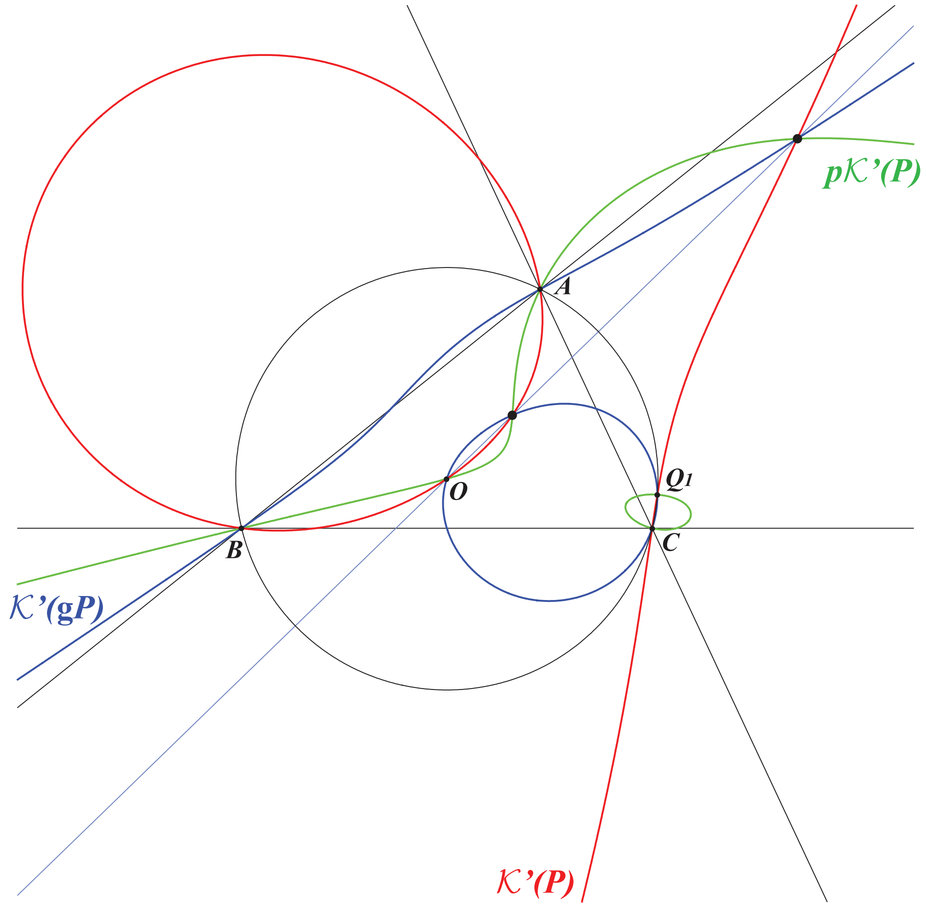

Figure 2: The pencil $\mathcal{K}'(P)$, $\mathcal{K}'(\mathbf{g}P)$

3.2 Other points on the cubic

Recall that $\mathcal{K}'(P)$ contains O, P, $\mathbf{ga}P$, $\mathbf{ig}P$, $\mathbf{igag}P$, E_1.

- The real infinite point $\infty P'$ of $\mathcal{K}'(P)$ is that of the line passing through P and the isoconjugate of the orthocenter H of ABC in the isoconjugation with fixed point P. $\infty P'$ is equivalently the crosspoint of P and $\mathbf{gig}P$.
- The sixth common point Q_1 of $\mathcal{K}'(P)$ and $(\mathcal{O})$ is the isogonal conjugate of the infinite point of the line passing through P and $\mathbf{g}P$.
- The poles A', B', C' of the sidelines BC, CA, AB in the circles PBC, PCA, PAB respectively. These circles have centers O_a, O_b, O_c. A' is the antipode of O_a on the circle BCO_a.
- $A'' = BC' \cap B'C$, B'' and C'' likewise.
- Y common point of the circles $A'BC$, $B'CA$, $C'AB$. Note that Y lies on the circle $O_aO_bO_c$ whose center is a point on the line $P, \mathbf{ig}P$.
- Y_1 common point of the circles $AB'C'$, $BC'A'$, $CA'B'$,
  Y_2 common point of the circles $AB''C''$, $BC''A''$, $CA''B''$,

Y_3 common point of the circles $A'B''C''$, $B'C''A''$, $C'A''B''$,
Y_4 common point of the circles $A''B'C'$, $B''C'A'$, $C''A'B'$.

Note that $\mathbf{ig}P$ is the common point of the circles $A''BC$, $B''CA$, $C''AB$.

- $A_0 = A, Q_1 \cap Y, A'$ and B_0, C_0 likewise.
- $U = B, C \cap A_0, \infty P'$ and V, W likewise. These are the remaining points on the sidelines of ABC.
- $A_1 = A, P \cap O, A'$ and B_1, C_1 likewise.
- $A_2 = A, \mathbf{ig}P \cap A', \infty P'$ and B_2, C_2 likewise.
- $A_2 = AQ_1 \cap YA'$, B_2 and C_2 likewise. A_2 is the coresidual of B, C and J_1, J_2, the circular points at infinity.
- $Q_2 = P, \infty P' \cap O$, $\mathbf{ig}P \cap YE_1$.
- $Q_3 = \mathbf{ig}P$, $\mathbf{ga}P \cap P$, $\mathbf{igag}P$.
- $Q_4 = P$, $\mathbf{ig}P \cap O$, $\mathbf{igag}P \cap \mathbf{ga}P, \infty P' \cap E_1Q_1$.
- Q_5, the third point on O, P.
- Q_6, the third point on P, $\mathbf{ga}P$.
- $Q_7 = P, Q_1 \cap Y$, $\mathbf{ga}P$.
- Q_8, the third point on O, $\mathbf{ga}P$.

3.3 Special cubics $\mathcal{K}'(P)$

Proposition 7. *$\mathcal{K}'(P)$ is a focal cubic if and only if P lies on the Napoleon cubic* **K005**.

See table 1 for examples of these cubics.

Proposition 8. *$\mathcal{K}'(P)$ is a pivotal cubic if and only if P lies on the second Brocard cubic* **K018**. *In this case, its pole lies on the Brocard axis, its pivot lies on the Euler line, its isopivot lies on* **K039**. *Furthermore, two points P on* **K018** *collinear with K give the same cubic.*

See table 2 for examples of these cubics.
Recall that **K039** is the Jerabek strophoid, the inversive image (in the circumcircle) of the Jerabek hyperbola.

4 The cubics $\mathcal{K}''(P)$

4.1 Properties of $\mathcal{K}''(P)$

Proposition 9. *The isogonal transform of $\mathcal{K}''(P)$ is the orthopivotal cubic $\mathcal{O}(\mathbf{g}P)$ or, equivalently, $\mathcal{K}''(P)$ is the isogonal transform of $\mathcal{O}(\mathbf{g}P)$.*

See [4] for details on orthopivotal cubics.

Corollary 1. *$\mathcal{K}''(P)$ contains a given point Q if and only if P lies on the circumconic passing through Q and* $\mathbf{gog}P$.

Indeed, the orthopivotal cubic $\mathcal{O}(P)$ contains Q if and only if P lies on the line Q, $\mathbf{o}Q$ hence the orthopivotal cubic $\mathcal{O}(\mathbf{g}P)$ contains Q if and only if P lies on the circumconic passing through $\mathbf{g}Q$ and $\mathbf{go}Q$.

Corollary 2. *The singular focus F'' of $\mathcal{K}''(P)$ is the inverse of that of the orthopivotal cubic $\mathcal{O}(\mathbf{g}P)$. See [4], §5.*

There are several equivalent characterizations of the singular focus F_P of $\mathcal{O}(P)$, not all of them mentioned in [4], §5. Here is a list of them :

1. If F_1 and F_2 are the real foci of the Steiner inellipse, then F_P is the reflection about the line F_1F_2 of the inverse of P in the circle with diameter F_1F_2.
2. F_P is the pole of P in the pencil of rectangular hyperbolas passing through the four foci of the Steiner inellipse.
3. F_P is the center of the rectangular hyperbola which is the polar conic of P with respect to the McCay cubic **K003** or the Kjp cubic **K024** and, more generally, with respect to any equilateral cubic of the pencil generated by these two latter cubics.

Corollary 3. *$\mathcal{K}''(P)$ decomposes if and only if $\mathcal{O}(\mathbf{g}P)$ itself decomposes.*

It follows that there are exactly eleven decomposed $\mathcal{K}''(P)$ and, in particular, $\mathcal{K}''(K)$ is the union of the Brocard axis OK and $(\mathcal{O})$. See [4], §6.1.

4.2 Other points on the cubic

Recall that $\mathcal{K}''(P)$ is a circular circumcubic passing through the isodynamic points X_{15} and X_{16}.

- The real infinite point $\infty P''$ of $\mathcal{K}''(P)$ is that of the line passing through P and $\mathbf{gt}P$. This line is actually the orthic line of the cubic $\mathcal{K}''(P)$.
- The sixth common point R_1 of $\mathcal{K}''(P)$ and $(\mathcal{O})$ is the isogonal conjugate of the infinite point of the line passing through the centroid G and $\mathbf{g}P$.
- The third common point R_2 with the Brocard axis OK is its second intersection with the circumconic passing through P, the Lemoine point K and R_1. R_2 lies on the lines $P, \mathbf{gt}P$ and X_{110}, R_1.
- The sixth point R_4 on the circumconic passing through X_{15} and X_{16} also lies on the circumconic passing through P and X_{523}. Hence R_4 is the isogonal conjugate of the intersection of the Fermat line and the line through X_{110} and $\mathbf{g}P$.

  R_4 is the coresidual of R_1, R_2 and the circular points at infinity J_1, J_2.

  P is the coresidual of X_{15}, X_{16}, J_1, J_2.
- The lines R_1, R_2 and $R_4, \infty P''$ meet at R_3 on $\mathcal{K}''(P)$. R_3 is the coresidual of A, B, C, P.

Proposition 10. *The orthic line of $\mathcal{K}''(P)$ passes through P, $\infty P''$ and R_2.*

This means that the polar conic of any point on this line is a rectangular hyperbola. It follows that the real asymptote of $\mathcal{K}''(P)$ is the homothetic of this orthic line under $h(F'', 2)$.

4.3 Special cubics $\mathcal{K}''(P)$

Proposition 11. *$\mathcal{K}''(P)$ is a focal cubic if and only if P lies on the second Brocard cubic* **K018***.*

In this case, the singular focus is the inverse of the singular focus of $\mathcal{O}(\mathbf{g}P)$ hence it lies on the bicircular quartic which is the inversive image of **K018**.

In particular, $\mathcal{K}''(X_{15})$ and $\mathcal{K}''(X_{16})$ are two strophoids, the isogonal transforms of **K061a** and **K061b**.

See table 2 for other examples of these cubics.

Proposition 12. *$\mathcal{K}''(P)$ is a pivotal cubic if and only if P lies on the Napoleon cubic* **K005***. In this case, the pivot lies on the Neuberg cubic* **K001***, the isopivot on* **K073** *and the pole on $p\mathcal{K}(X_6 \times X_{50}, X_6)$.*

See table 1 for examples of these cubics.

The pivot is the intersection of the lines OP and $H\mathbf{g}P$. If P and $\mathbf{g}P$ are two points on **K005** (hence collinear with X_5) then the corresponding pivots of $\mathcal{K}''(P)$ and $\mathcal{K}''(\mathbf{g}P)$ are two isogonal conjugate points on **K001** on a parallel to the Euler line of ABC.

Recall that **K073** is $p\mathcal{K}(X_{50}, X_3)$, the inversive image of the Neuberg cubic **K001** in the circumcircle. The isopivot of $\mathcal{K}''(P)$ is the inverse of the isogonal conjugate of its pivot.

$p\mathcal{K}(X_6 \times X_{50}, X_6)$ is the pivotal isocubic with pivot the Lemoine point X_6 and isopivot X_{50}, the barycentric product of X_{15} and X_{16}. It is also the isogonal transform of $p\mathcal{K}(X_{94}, X_{94})$. The line passing through the poles of $\mathcal{K}''(P)$ and $\mathcal{K}''(\mathbf{g}P)$ contains a fixed point namely the barycentric product of X_{30} and X_{186}. This point is the intersection of the lines X_4X_{1989}, X_6X_{74}, $X_{25}X_{1576}$, $X_{30}X_{1990}$, $X_{50}X_{186}$, $X_{53}X_{3018}$, $X_{526}X_{2081}$, etc.

Another remarkable thing to observe is that the three lines passing through the pivots, the isopivots and the poles of $\mathcal{K}''(P)$ and $\mathcal{K}''(\mathbf{g}P)$ are concurrent at a point lying on the circum-hyperbola passing through X_{30}, X_{74}, X_{186}, X_{526}, X_{1464}, X_{1511}. This is the isogonal transform of the real asymptote of the Neuberg cubic.

Proposition 13. *$\mathcal{K}''(P)$ is a nonpivotal isocubic if and only if P lies on* **K397***, the isogonal nonpivotal isocubic $n\mathcal{K}(X_6, X_{30}, X_2)$.*

K397 contains X_2, X_6, X_{110}, X_{523} giving the cubics **K018** $= \mathcal{K}''(X_2)$, **K148** $= \mathcal{K}''(X_{110})$. This latter cubic **K148** is $n\mathcal{K}_0(X_{50}, X_6)$, the isogonal transform of **K064** $= n\mathcal{K}_0(X_{1989}, X_{1989})$, the orthopivotal cubic with orthopivot X_{523}.

Proposition 14. *$\mathcal{K}''(P)$ is an orthopivotal cubic if and only if P lies on the Kiepert hyperbola.*

Its orthopivot is the harmonic conjugate of $\mathbf{g}P$ with respect to O and K. In this case, $\mathcal{K}''(P)$ contains the Fermat points X_{13} and X_{14}.

Two points P and P' on the Kiepert hyperbola collinear with K give two cubics $\mathcal{K}''(P)$ and $\mathcal{K}''(P')$ such that one is the isogonal transform of the other.

5 Tables and special cases

5.1 P on the Napoleon cubic K005

$\mathcal{K}(P)$ and $\mathcal{K}'(P)$ are focal cubics, $\mathcal{K}''(P)$ is a pivotal cubic.

$\mathcal{L}^\infty$ is the line at infinity, $\mathcal{J}$ is the Jerabek hyperbola.

Table 1: Cubics with P on the Napoleon cubic **K005**

P	$\mathcal{K}(P)$	$\mathcal{K}'(P)$	$\mathcal{K}''(P)$
X_1	**K086**	$(\mathcal{O}) \cup OI$	**K206**
X_3	$\mathcal{L}^\infty \cup \mathcal{J}$	**K039**	**K114**
X_4	$(\mathcal{O}) \cup OG$	$\mathcal{L}^\infty \cup \mathcal{J}$	**K001**
X_5	**K464**	**K465**	**K050**
X_{17}	X_{13}, X_{16}, X_{17}, X_{61}	X_3, X_{13}, X_{16}, X_{17}, X_{622}	**K261a**
X_{18}	X_{14}, X_{15}, X_{18}, X_{62}	X_3, X_{14}, X_{15}, X_{18}, X_{621}	**K261b**
X_{54}	**K467**	X_3, X_{54}, X_{186}, X_{2070}	**K073**

5.2 P on the second Brocard cubic K018

$\mathcal{K}(P)$ and $\mathcal{K}'(P)$ are pivotal cubics, $\mathcal{K}''(P)$ is a focal cubic.

Recall that two points P on **K018** collinear with G (resp. K) give the same cubic $\mathcal{K}(P)$ (resp. $\mathcal{K}'(P)$).

Table 2: Cubics with P on the second Brocard cubic **K018**

P	$\mathcal{K}(P)$	$\mathcal{K}'(P)$	$\mathcal{K}''(P)$
X_2	**K273**	**K043**	**K018**
X_6	**K043**	**K108**	$(\mathcal{O}) \cup OK$
X_{13}	**K261a**	**K001**	**K262b**
X_{14}	**K261b**	**K001**	**K262a**
X_{15}	**K261b**	**K073**	**gK061a**
X_{16}	**K261a**	**K073**	**gK061b**
X_{111}	**K273**	**K108**	**gK065**

gK061a and **gK061b** are the isogonal transforms of **K061a** and **K061b**. These are two strophoids.

gK065 contains X_6, X_{15}, X_{16}, X_{23}, X_{111}, X_{2854}. It is the isogonal transform of the central focal orthopivotal cubic **K065**.

5.3 P on the Thomson cubic K002

When P is a point on the Thomson cubic **K002**, the three cubics $\mathcal{K}(P)$, $\mathcal{K}'(P)$ and $\mathcal{K}''(P)$ – of the same pencil $\mathcal{F}(P)$ – meet $(\mathcal{O})$ at the same points namely A, B, C, the circular points at infinity and the isogonal conjugate of the infinite point of the lines $P, \mathbf{g}P$ or $P, \mathbf{ag}P$.

Note that the three corresponding cubics $\mathcal{K}(\mathbf{g}P)$, $\mathcal{K}'(\mathbf{g}P)$ and $\mathcal{K}''(\mathbf{g}P)$ contain these same six points on $(\mathcal{O})$.

The two last base points of $\mathcal{F}(P)$, apart the seven already mentioned, are collinear with P. See table 3.

5.4 Other remarkable cubics

5.4.1 P on the Jerabek hyperbola $\mathcal{J}$

$\mathcal{K}(P)$ contains the circumcenter O if and only if P lies on the Jerabek hyperbola. In this case, $\mathcal{K}(P)$, $\mathcal{K}'(P)$, $\mathcal{K}''(P)$ all contain O. Furthermore, $\mathcal{K}''(P)$ also passes through X_{74}.

Table 3: Cubics with P on the Thomson cubic **K002**

P	$\mathcal{K}(P)$ or X_i for $i =$	$\mathcal{K}'(P)$ or X_i for $i =$	$\mathcal{K}''(P)$ or X_i for $i =$
X_1	**K086**	$(\mathcal{O}) \cup OI$	**K206**
X_2	**K273**	**K043**	**K018**
X_3	$\mathcal{L}^\infty \cup \mathcal{J}$	**K039**	**K114**
X_4	$(\mathcal{O}) \cup OG$	$\mathcal{L}^\infty \cup \mathcal{J}$	**K001**
X_6	**K043**	**K108**	$(\mathcal{O}) \cup OK$
X_9	$9, 57, 84, 527, 2291$	$3, 9, 55, 2078, 2291$	$1, 9, 15, 16, 284, 1795, 2291, 3065$
X_{57}	$9, 40, 57, 527, 2078, 2291$	$3, 57, 1155, 1436, 2291$	$15, 16, 57, 284, 2291$
X_{223}	$223, 282$	$3, 198, 223$	$15, 16, 223$
X_{282}	$223, 282, 1490$	$3, 282$	$15, 16, 282$

5.4.2 P on the Kiepert hyperbola $\mathcal{K}$

$\mathcal{K}''(P)$ contains the Fermat points X_{13}, X_{14} if and only if P lies on the Kiepert hyperbola. In this case, $\mathcal{K}''(P)$ is an orthopivotal cubic with orthopivot on the Brocard axis. These are the cubics of the pencil generated by the Neuberg cubic **K001** and the second Brocard cubic **K018**.

Note that $\mathcal{K}(P)$ and $\mathcal{K}'(P)$ contain X_{13} and X_{16} if and only if P lies on **K261a**. They contain X_{14} and X_{15} if and only if P lies on **K261b**.

5.4.3 P on the circumcircle $(\mathcal{O})$ or on the Euler line $(\mathcal{E})$

$\mathcal{K}(P)$ contains the orthocenter H if and only if P lies on the circumcircle $(\mathcal{O})$ of ABC or on the Euler line $(\mathcal{E})$ of ABC.

- When P lies on $(\mathcal{O})$,
  1. the infinite point of $\mathcal{K}(P)$ is $\mathbf{g}P$ and the real asymptote of $\mathcal{K}(P)$ is perpendicular to the Simson line of P,
  2. the tangent at P to $\mathcal{K}(P)$ passes through O,
  3. the tangents at A, B, C to $\mathcal{K}(P)$ concur at the Lemoine point K,
  4. the singular focus of $\mathcal{K}(P)$ is the reflection of O about P hence this lies on the circle $\mathcal{C}(O, 2R)$,
  5. the three cubics obviously meet $(\mathcal{O})$ at the same points. Recall that this is also true when P lies on the Thomson cubic **K002**.

- When P lies on $(\mathcal{E})$, the three cubics share the same (real) point at infinity, that of the second tangent – apart $(\mathcal{E})$ – drawn from P to the inscribed conic whose perspector is the isotomic conjugate of the center X_{125} of the Jerabek hyperbola. This conic has center the complement of X_{125} and contains X_4, X_{69}, X_{1092}, X_{1974}, X_{3043}.

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Table 4: $\mathcal{K}(P)$ with P on $(\mathcal{O})$

P on $(\mathcal{O})$	centers on $\mathcal{K}(P)$	$\mathcal{K}(P)$
X_{74}	X_3, X_4, X_{30}, X_{74}	
X_{98}	X_4, X_{76}, X_{98}, X_{511}	
X_{104}	X_4, X_8, X_{104}, X_{517}	
X_{106}	X_1, X_4, X_{106}, X_{519}, X_{1168}, X_{1320}	
X_{107}	X_4, X_{107}, X_{250}, X_{520}	
X_{108}	X_4, X_{59}, X_{108}, X_{521}	
X_{111}	X_2, X_4, X_6, X_{23}, X_{111}, X_{524}, X_{671}, X_{895}	**K273**
X_{112}	X_4, X_{112}, X_{249}, X_{525}	
X_{733}	X_4, X_{83}, X_{732}, X_{733}, X_{1916}	
X_{759}	X_4, X_{21}, X_{58}, X_{80}, X_{758}, X_{759}, X_{1325}	**K306**
X_{1141}	X_4, X_{54}, X_{265}, X_{1141}, X_{1154}	
X_{2291}	X_4, X_9, X_{57}, X_{527}, X_{1156}, X_{2291}	
X_{2373}	X_4, X_{69}, X_{316}, X_{1177}, X_{2373}, X_{2393}	
X_{2380}	X_4, X_{14}, X_{15}, X_{532}, X_{1337}, X_{2380}	
X_{2381}	X_4, X_{13}, X_{16}, X_{533}, X_{1338}, X_{2381}	

Table 5: $\mathcal{K}'(P)$ with P on $(\mathcal{O})$

P on $(\mathcal{O})$	centers on $\mathcal{K}'(P)$	$\mathcal{K}'(P)$
X_{74}	X_3, X_{64}, X_{74}, X_{186}, X_{399}	
X_{98}	X_3, X_{98}, X_{1503}, X_{1676}, X_{1677}, X_{1691}, X_{2353}	
X_{104}	X_3, X_{104}, X_{1319}, X_{2932}	
X_{106}	X_3, X_{36}, X_{56}, X_{106}, X_{2390}	
X_{111}	X_3, X_6, X_{23}, X_{25}, X_{111}, X_{187}, X_{1177}, X_{2393}, X_{2930}	**K108**
X_{112}	X_3, X_{112}, X_{512}, X_{2079}	
X_{733}	X_3, X_{32}, X_{733}, X_{2076}	
X_{759}	X_1, X_3, X_{759}, X_{1324}, X_{2217}, X_{2948}	
X_{1141}	X_3, X_4, X_{1141}, X_{2070}	
X_{1299}	X_3, X_{155}, X_{1299}, X_{2931}	
X_{1300}	X_3, X_{24}, X_{30}, X_{1300}	
X_{2291}	X_3, X_{55}, X_{1436}, X_{2078}, X_{2291}	
X_{2373}	X_3, X_{22}, X_{67}, X_{468}, X_{2373}, X_{2936}	

Table 6: $\mathcal{K}''(P)$ with P on $(\mathcal{O})$

X_{98}	X_{13}, X_{14}, X_{15}, X_{16}, X_{98}, X_{182}, X_{542}	**K292**
X_{104}	X_{1}, X_{15}, X_{16}, X_{104}, X_{2771}, X_{3065}	
X_{106}	X_{15}, X_{16}, X_{58}, X_{106}, X_{2842}	
X_{111}	X_{6}, X_{15}, X_{16}, X_{23}, X_{111}, X_{2854}	**gK065**
X_{477}	X_{15}, X_{16}, X_{30}, X_{477}, X_{1138}	
X_{1141}	X_{4}, X_{15}, X_{16}, X_{567}, X_{1141}, X_{1263}	
X_{1297}	X_{15}, X_{16}, X_{1297}, X_{1350}, X_{2781}	
X_{2770}	X_{2}, X_{15}, X_{16}, X_{524}, X_{2770}	

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