McCay Stelloids

Bernard Gibert

Created : February 21, 2014
Last update : October 23, 2015

Abstract

We study the stelloids similar to the McCay cubic and also several pencils generated by some of them. We give several geometric constructions related to these cubics and in particular that of their Hessian cubic.

1 Introduction

The McCay cubic K003 is probably one of the most interesting stelloids since it presents an important number of geometric properties. See [3].

Recall that a cubic is said to be a stelloid when it has three real concurring asymptotes making 60° angles with one another. These are denoted by $\mathcal{K}_{60}^{+}$ in [1]. Equivalently, it is a cubic whose Laplacian identically vanishes or such that the polar conic of every point in the plane is a rectangular hyperbola. See [1], chapter 5, for further details.

In this paper, we are interested in circum-stelloids having the same asymptotic directions as the McCay cubic i.e. having three real asymptotes perpendicular to the sidelines of the Morley triangle and concurring at a certain (finite) point called the "**radial center**" of the stelloid. These are hereby called McCay stelloids.

Recall that the asymptotes of the McCay cubic concur at the centroid G of the reference triangle ABC. To construct these asymptotes, first draw the Stammler hyperbola (passing through the in/excenters and O, K, etc, whose center is X_{110}) then draw its image under $h(O, 1/2)$ to obtain another rectangular hyperbola meeting the circumcircle at X_{110} and three other points. The parallels at G to the lines passing through O and these points are the asymptotes of the McCay cubic.

Since [3] contains a good number of such McCay stelloids, we also intend to give a classification according to the type of the cubic. See §8 for a list of McCay stelloids.

2 Definition and first properties

The Laplacian of a circum-cubic is a polynomial of degree 1 which identically vanishes when three conditions (on the coefficients of the cubic) are fulfilled. In such case, the three asymptotes concur and when we impose that the point of concurrence be the given (finite) point X, we obtain three more conditions. It is sufficient to express that the polar conic of X splits into the line at infinity and another line.

On the other hand, this same circum-cubic coincide at infinity with the McCay cubic when three other conditions are fulfilled.

This gives a system of nine equations which has always a unique solution hence there is a unique such circum-cubic.

Proposition 1 *If $X = u : v : w$ is a given finite point, there is one and only one circum-stelloid $\mathcal{K}(X)$ with radial center X and asymptotes parallel to those of the McCay cubic.*

It is known that any line passing through X meets $\mathcal{K}(X)$ at three points whose isobarycenter is X. Thus, if X lies on $\mathcal{K}(X)$, the cubic must be a central cubic denoted $\mathcal{K}_{60}^{++}$ in [1]. See §4.1.

2.1 Related curves and equations of $\mathcal{K}(X)$

Recall that $\mathcal{K}(G)$ is the McCay cubic itself. Now, if $X \neq G$, let us denote by

- $\mathcal{L}(X)$ the parallel at O to the line GX,
- $\mathcal{H}(X)$ the rectangular circum-hyperbola which is the isogonal transform of $\mathcal{L}(X)$,
- $\mathcal{C}(X)$ the isogonal pivotal cubic $p\mathcal{K}(X_6, Y)$ with pivot Y, the homothetic of X under $h(X_5, 3)$, a point obviously lying on $\mathcal{L}(X)$ since $h(X_5, 3)$ maps G onto O.
- $\mathcal{M}$ the McCay cubic K003.

After some easy computations, we find that the equation of $\mathcal{K}(X)$ can be written

$$\mathcal{K}(X) = (u + v + w)\ \mathcal{M} - (x + y + z)\ \mathcal{H}(X), \tag{1}$$

or equivalently

$$\mathcal{K}(X) = \mathcal{C}(X) - (a^2yz + b^2zx + c^2xy)\ \mathcal{L}(X). \tag{2}$$

Let $\mathcal{K}^*(X)$ be the isogonal transform of $\mathcal{K}(X)$. $\mathcal{K}^*(X)$ is a circum-cubic passing through O and the vertices N_1, N_2, N_3 of the circum-normal triangle, these points obviously lying on the McCay cubic.

The equation of $\mathcal{K}^*(X)$ can be written

$$\mathcal{K}^*(X) = (u + v + w)\ \mathcal{M} + (a^2yz + b^2zx + c^2xy)\ \mathcal{L}(X), \tag{3}$$

or equivalently

$$\mathcal{K}^*(X) = \mathcal{C}(X) + (x + y + z)\ \mathcal{H}(X). \tag{4}$$

2.2 Consequences and properties of $\mathcal{K}(X)$

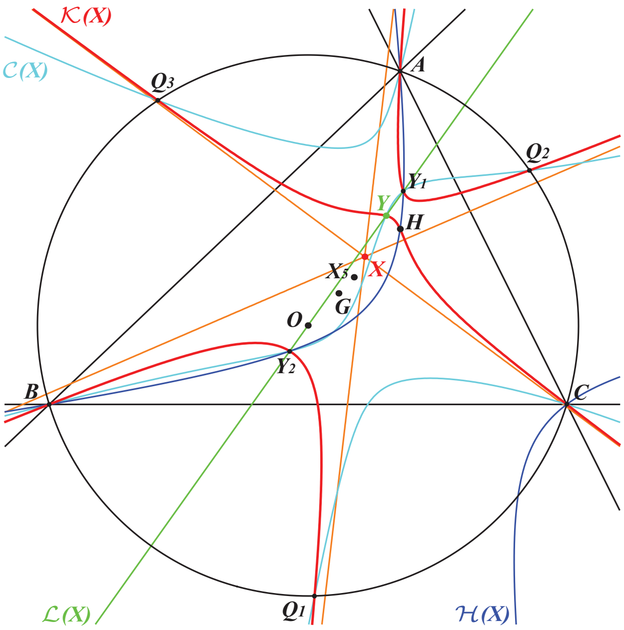

Figure 1: $\mathcal{K}(X)$ with $\mathcal{L}(X)$, $\mathcal{H}(X)$, $\mathcal{C}(X)$

By equation (1), we find that

1. $\mathcal{K}(X)$ always contains the orthocenter H of ABC since $\mathcal{M}$ and $\mathcal{H}(X)$ both contain H.

2. $\mathcal{K}(X)$ meets the McCay cubic at seven points independent of X namely their three common points at infinity, A, B, C, H and also two other points Y_1, Y_2 on the hyperbola $\mathcal{H}(X)$.

   By equation (2), we also find that

3. $\mathcal{K}(X)$ meets the circumcircle (O) of ABC at the same points as $\mathcal{C}(X)$ namely A, B, C and three other (real or not) points Q_1, Q_2, Q_3.

4. $\mathcal{K}(X)$ and $\mathcal{C}(X)$ meet at three other points which must lie on the line $\mathcal{L}(X)$ and one of them is Y, the pivot of $\mathcal{C}(X)$. It follows that the two other points must be isogonal conjugates hence they must also lie on $\mathcal{H}(X)$. In other words, the points Y_1, Y_2 are the common points of $\mathcal{L}(X)$ and $\mathcal{H}(X)$. See figure 1.

5. Since Y_1, Y_2 also lie on $\mathcal{K}^*(X)$, the cubics $\mathcal{K}(X)$ and $\mathcal{K}^*(X)$ have four remaining common points which are the foci of the inscribed conic with center H', the homothetic of H under $h(X, -1/2)$ and the midpoint of OY, a point clearly on the line $\mathcal{L}(X)$. Let us denote by F_1, F_2 the two real foci and by F_1', F_2' the two imaginary conjugate foci.

   Indeed, by equations (2) and (3), we have

$$\mathcal{K}(X) + \mathcal{K}^*(X) = (u+v+w)\ \mathcal{M} + \mathcal{C}(X) = p\mathcal{K}(X_6, H') \tag{5}$$

   and, similarly, by equations (2) and (4), we have

$$\mathcal{K}^*(X) - \mathcal{K}(X) = (x+y+z)\ \mathcal{H}(X) + (a^2yz + b^2zx + c^2xy)\ \mathcal{L}(X) = \mathcal{N}(X) \tag{6}$$

   which is in fact the non-pivotal isogonal focal cubic $\mathcal{N}(X)$, the locus of foci of inscribed conics with center on the line $\mathcal{L}(X)$. Its singular focus F is the isogonal conjugate of the infinite point of $\mathcal{L}(X)$. See figure 2.

Note that the polar conic of F in $\mathcal{N}(X)$ is the circle passing through F, Y_1, Y_2.

Remark : the cubic with equation (7) very similar to equation (6)

$$(x+y+z)\ \mathcal{H}(X) - (a^2yz + b^2zx + c^2xy)\ \mathcal{L}(X) \tag{7}$$

is the circular isogonal pivotal cubic whose pivot is the infinite point of the line GX.

2.3 Construction of the points Q_1, Q_2, Q_3

We simply adapt and complete a construction of [6]. See figure 3.

The polar conic C_Y of Y in $\mathcal{C}(X)$ is the diagonal rectangular hyperbola passing through Y and the four in/excenters of ABC. Its center ω lies on (O). The homothety $h(Y, 1/2)$ transforms C_Y into H_Y passing through ω and the points Q_1, Q_2, Q_3. Recall that Y is the orthocenter of $Q_1Q_2Q_3$. See [6] for further informations.

These points Q_1, Q_2, Q_3 are all real and distinct (i.e. forming a proper triangle) if and only if X lies inside the deltoid Δ with center G whose axes are the asymptotes of the McCay cubic. Its cusps lie on the circle with center G whose radius is that of the circumcircle (O) of ABC. See figure 4.

When X lies on Δ, H_Y is tangent to (O) hence $\mathcal{K}(X)$ is also tangent to (O). In particular, if X is one of the cusps of Δ then the three points Q_1, Q_2, Q_3 coincide and $\mathcal{K}(X)$ has a flex at this point.

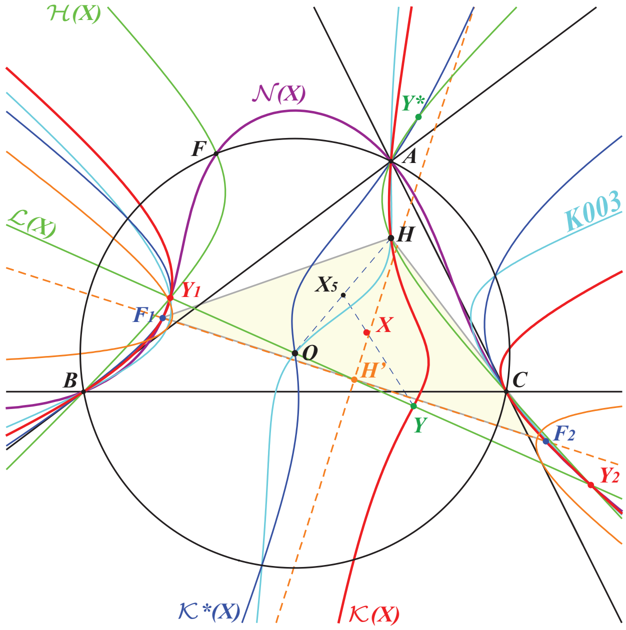

Figure 2: $\mathcal{K}(X)$ with $\mathcal{K}^*(X)$

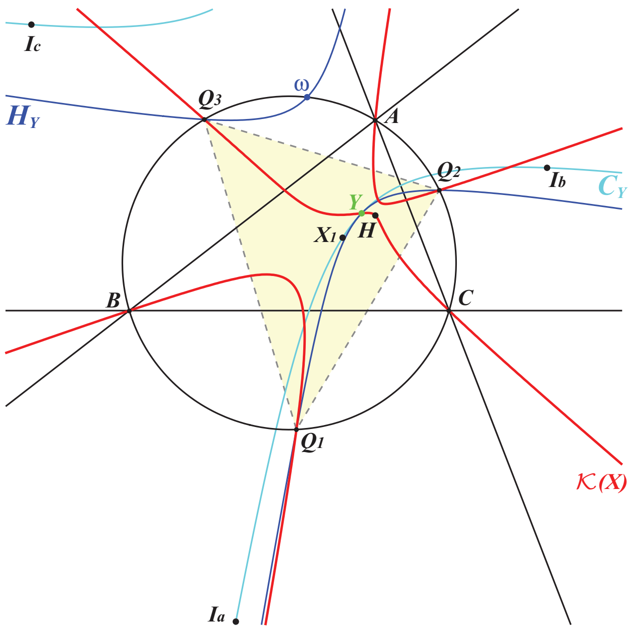

Figure 3: Construction of the points Q_1, Q_2, Q_3

2.4 Construction of the cubics $\mathcal{K}(X)$, $\mathcal{K}^*(X)$ and other points on $\mathcal{K}(X)$

Recall that Y is the homothetic of the radial center X under $h(X_5, 3)$.

Draw two parallels L_O, L_Y passing through O, Y respectively and let L_O^* be the rectangular circum-hyperbola which is the isogonal transform of L_O. L_Y and L_O^* meet at two points lying on $\mathcal{K}(X)$ which is actually a special case of cubics of the class CL055 in [3]. The coresidual of A, B, C, H is clearly Y.

Obviously, L_O and L_O^* meet at two points lying on the McCay cubic. See figure 5.

$\mathcal{K}^*(X)$ is obtained similarly. It is sufficient to intersect L_O and L_Y^*.

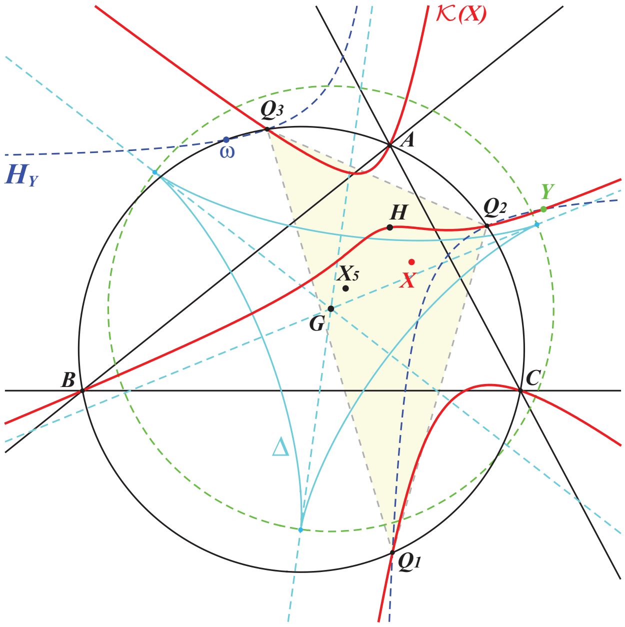

Figure 4: The deltoid Δ and the points Q_1, Q_2, Q_3

If the line L_O contains A, L_O^* splits into the altitude AH and the sideline BC which meet L_Y at A_1 and U respectively. The corresponding points B_1, V and C_1, W are defined likewise.

If the line L_O is the perpendicular bisector of BC, L_O^* contains the antipode of A on (O) and meets the perpendicular at Y to BC at two other points A_2, A_3 on $\mathcal{K}(X)$. The four other points B_2, B_3 and C_2, C_3 are defined likewise. These six points are not always real.

If the line L_Y is the line HY, the hyperbola L_O^* meets L_Y at Y', the third point of $\mathcal{K}(X)$ on the line HY. This point Y' lies on the nodal cubic K028 and it is in fact the nineth common point of $\mathcal{K}(X)$ and K028 since H must be counted twice.

When $X \neq X_5$ i.e. when $\mathcal{K}(X)$ is not the central cubic K026 , if the line L_Y is the line X_5XY then L_Y meets $\mathcal{K}(X)$ at Y and two other points Z_1, Z_2 whose midpoint is X_5. Indeed, any line passing through the radial center X of a stelloid meets this stelloid at three points whose isobarycenter is X. If Y is one of these points, the midpoint of the others must be X since X is at one third from X_5 to Y.

These two points Z_1, Z_2 are not necessarily real. This depends of the position of X with regard to the eight regions delimited by the Euler line X_5X_{51} of the orthic triangle and the three parallels at X_5 to the asymptotes of the McCay cubic K003. When X lies inside one of the four regions that contains one the points A, B, C, O, H then the points are real. See figure 6 where these four regions are represented in yellow.

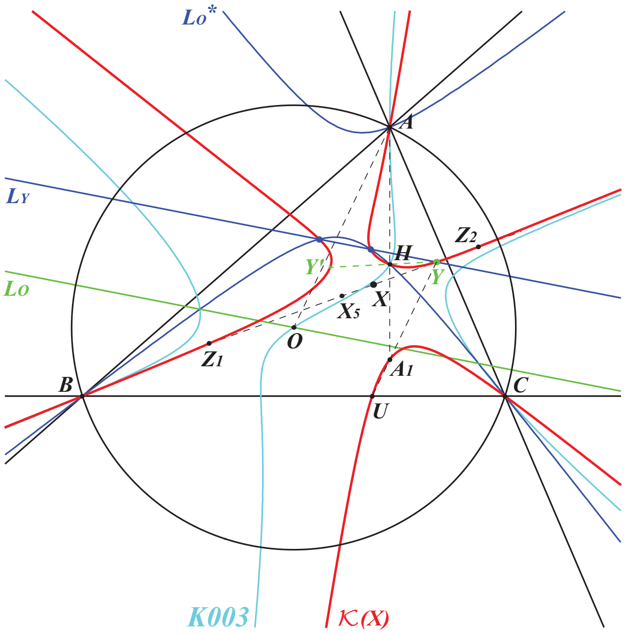

Figure 5: Construction of $\mathcal{K}(X)$

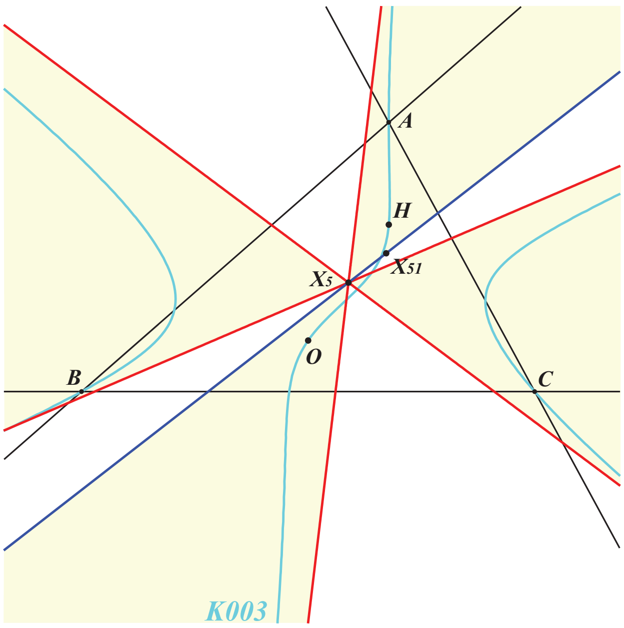

Figure 6: Reality of the points Z_1, Z_2

2.5 Another characterization of $\mathcal{K}(X)$ and construction of the cubics $\mathcal{K}(X)$ passing through a prescribed point

Let M be a variable point and denote by M' the isogonal conjugate of M with respect to the pedal triangle of M. Recall that Y is the homothetic of X under $h(X_5, 3)$. A direct computation gives

Proposition 2 *$\mathcal{K}(X)$ is the locus of M such that Y, M and M' are collinear.*

In particular, the McCay cubic can be seen as the locus of M such that O, M and M' are collinear.

It follows that any cubic $\mathcal{K}(X)$ passing through a given point P different of the seven points mentioned at §2.2 must contain another point Q depending uniquely of P. This point Q is the second intersection of rectangular circum-hyperbola passing through P with the line L_P passing through P and the isogonal conjugate P' of P with respect to the pedal triangle of P.

Now, if Y is a point on L_P and if X is its image under $h(X_5, 1/3)$, the construction of §2.4 above gives the cubic $\mathcal{K}(X)$ that passes through P and Q. When Y traverses L_P, these cubics belong to a same pencil since they have nine common points.

2.6 Polar conics

Recall that the polar conic in the stelloid $\mathcal{K}(X)$ of any point of the plane is a rectangular hyperbola and all these hyperbolas form a net of conics.

By §2.4 we have enough points on $\mathcal{K}(X)$ to construct the polar conic of several points on the curve namely A, B, C, H, Y.

2.6.1 Polar conics of A, B and C

Let us recall that the parallels at Y to the lines OC, OB, OA meet the lines AB, AC, AH at W, V, A_1 respectively.

Denote by A', V', W' the harmonic conjugates of A with respect to H and A_1, C and V, B and W respectively. The polar conic of A is the rectangular hyperbola $\mathcal{H}_A$ passing through A, A', V', W'. See figure 7.

The other polar conics $\mathcal{H}_B$, $\mathcal{H}_C$ are defined likewise and these three conics can be used to generate the net of polar conics since A, B, C are not collinear.

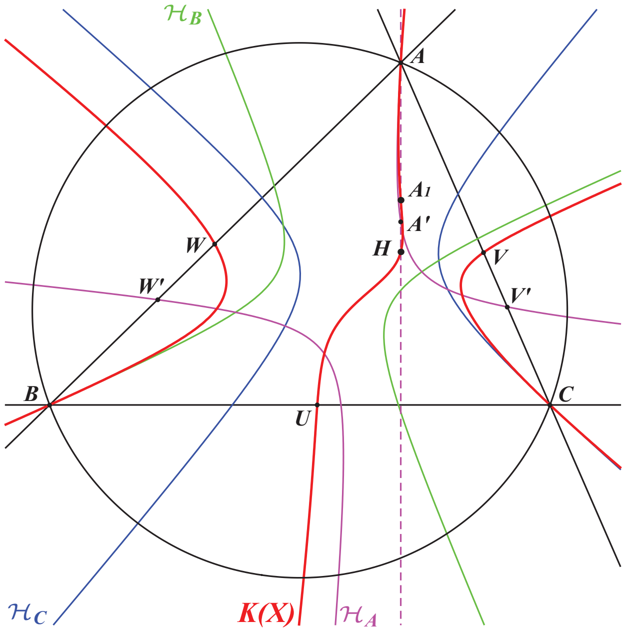

Figure 7: Polar conics $\mathcal{H}_A$, $\mathcal{H}_B$, $\mathcal{H}_C$ of A, B, C

Naturally, the tangents at A, B, C to $\mathcal{K}(X)$ are the tangents at the same points to $\mathcal{H}_A$, $\mathcal{H}_B$, $\mathcal{H}_C$. These tangents are generally not concurrent unless $\mathcal{K}(X)$ is a $ps\mathcal{K}$. See §4.4.

2.6.2 Polar conic of H

The polar conic $\mathcal{H}_H$ of H is the rectangular hyperbola passing through H and its harmonic conjugates with respect to A and A_1, B and B_1, C and C_1, Y and Y' respectively.

2.6.3 Polar conic of Y

The polar conic $\mathcal{H}_Y$ of Y is the rectangular hyperbola passing through Y and its harmonic conjugates with respect to U and A_1, V and B_1, W and C_1, H and Y' respectively.

When the points A_2, A_3, B_2, B_3, C_2, C_3 on $\mathcal{K}(X)$ are real, it is possible to use some (or all) of them to draw the polar conic of Y since it is the rectangular hyperbola passing through Y and its harmonic conjugates with respect to A_2 and A_3, B_2 and B_3, C_2 and C_3 respectively.

2.6.4 Polar conic of any point M on $\mathcal{K}(X)$ and tangent at M

Let us now consider a point M (different of the five points already mentioned above) on $\mathcal{K}(X)$ and let $\mathcal{H}_M$ be its polar conic.

The line MA meets $\mathcal{H}_A$ again at M_a and let M'_a be the harmonic conjugate of M with respect to A and M_a. Two other points M'_b, M'_c are defined likewise.

$\mathcal{H}_M$ is the rectangular hyperbola passing through M, M'_a, M'_b, M'_c and the harmonic conjugate of M with respect to Y and N as in §2.4.

Obviously the tangent at M to $\mathcal{K}(X)$ is the tangent at M to $\mathcal{H}_M$.

3 Hessian $\Sigma(X)$ of $\mathcal{K}(X)$ and related curves

Let us remind that the Hessian Σ of a cubic curve $\mathcal{K}$ is the locus of points M whose polar conic degenerates into two lines and then their intersection N is also a point on Σ. These two points are said to be conjugated points on Σ since the polar conic of N degenerates into two lines secant at M.

Now, if three non-collinear points are taken on $\mathcal{K}$, Σ is also the Jacobian of their polar conics in $\mathcal{K}$. In other words, M lies on Σ if and only if the polar lines of M in the three polar conics concur at N and vice-versa.

See the related involution Φ in §3.4 below.

3.1 Properties of $\Sigma(X)$

The Hessian $\Sigma(X)$ of $\mathcal{K}(X)$ is a focal cubic whose singular focus is the radial center X of $\mathcal{K}(X)$. It follows that the polar conic $\mathcal{P}(X)$ of X in $\Sigma(X)$ is a circle.

The orthic line $\mathcal{O}(X)$ of $\Sigma(X)$ is the locus of points whose polar conic in $\Sigma(X)$ is a rectangular hyperbola. This line is undefined if and only if $\Sigma(X)$ is at the same time a stelloid hence if and only if $\Sigma(X)$ splits into the line at infinity and a conic. More informations below in §4.2. Note that the polar conic of X in $\mathcal{K}(X)$ splits into the line at infinity and $\mathcal{O}(X)$. Hence $\mathcal{K}(X)$ meets $\mathcal{O}(X)$ at three (not necessarily all real) points where the tangents to $\mathcal{K}(X)$ concur at X. The images of these points under $h(X, -2)$ are three other points on $\mathcal{K}(X)$ since the isobarycenter of three points of $\mathcal{K}(X)$ lying on a line passing through X must be X.

The polar line $\mathcal{A}(X)$ of X in $\mathcal{K}(X)$ is the real asymptote of $\Sigma(X)$. It is the homothetic of $\mathcal{O}(X)$ under $h(X, 2)$.

On the other hand, the polar line of X in $\Sigma(X)$ is the tangent at X to $\Sigma(X)$ and to the circle $\mathcal{P}(X)$. This tangent contains the tangential Z of X in $\Sigma(X)$ and Z is the intersection of $\Sigma(X)$ with its real asymptote $\mathcal{A}(X)$.

The satellite line $\mathcal{S}(X)$ of $\mathcal{K}(X)$ is the line homothetic of $\mathcal{O}(X)$ under $h(X, 2/3)$. Recall that this line passes through the three intersections of $\mathcal{K}(X)$ and its three asymptotes. See figure 8.

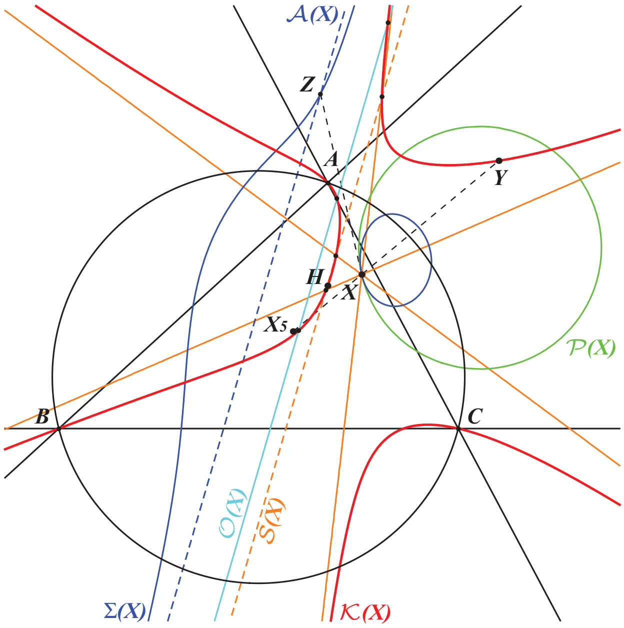

Figure 8: Hessian $\Sigma(X)$ of $\mathcal{K}(X)$

3.2 An important property

It is known that the polar lines of a point M (with respect to a cubic $\mathcal{K}$) with respect to the polar conics (in this same cubic) of all the points of the plane are concurrent (at N) if and only if M lies on the Hessian Σ of $\mathcal{K}$ and then N also lies on Σ. As already said, M and N are said to be conjugated points on Σ.

Applying this property to $\mathcal{K}(X)$, we obtain

Proposition 3 *The polar lines of X with respect to the polar conics (in $\mathcal{K}(X)$) of all the points in the plane (except X) are parallel. Furthermore, they are parallel to the orthic line $\mathcal{O}(X)$ of $\Sigma(X)$.*

This is obviously true for the polar conics $\mathcal{H}_A$, $\mathcal{H}_B$, $\mathcal{H}_C$ of A, B, C we met in §2.6.1. These polar lines Δ_A, Δ_B, Δ_C are represented in figure 9. Let us denote by ∞_X their common point at infinity.

Consequently, the polar conic of ∞_X in $\mathcal{K}(X)$ must decompose into two perpendicular lines secant at X since ∞_X lies on $\Sigma(X)$. These two lines are parallel to the asymptotes of the rectangular circum-hyperbola which is the isogonal transform of the parallel at O to the polar line of X in $\mathcal{K}(X)$.

Recall that the polar conic of X in $\mathcal{K}(X)$ splits into the line at infinity and the harmonic polar of X hence the polar line of X in this decomposed conic is undefined.

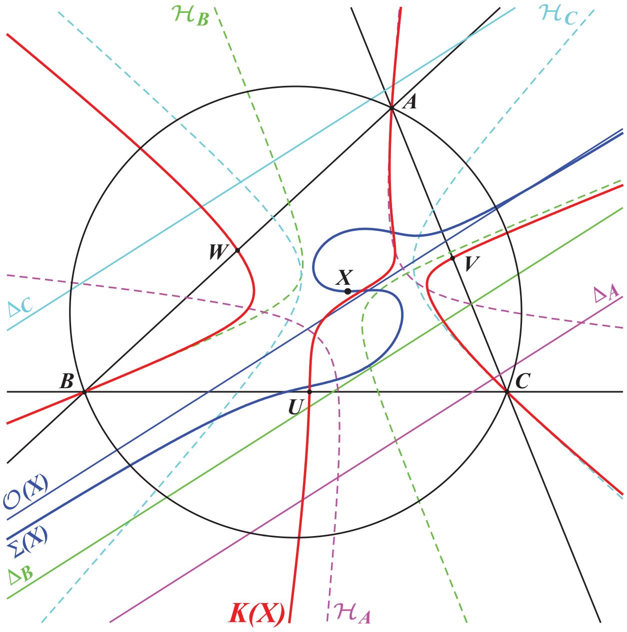

Figure 9: The polar lines Δ_A, Δ_B, Δ_C

3.3 Construction of $\Sigma(X)$

Let ℓ_X be a variable line passing through X and M a variable point on ℓ_X. Denote by δ_A, δ_B, δ_C the polar lines of M in $\mathcal{H}_A$, $\mathcal{H}_B$, $\mathcal{H}_C$.

δ_B and δ_C meet at E_A and, when M traverses ℓ_X, the locus of E_A is a conic γ_A passing through ∞_X (obtained when $M = X$) and the poles P_b, P_c of ℓ_X in $\mathcal{H}_B$, $\mathcal{H}_C$. To construct γ_A, it is sufficient to intersect the polar lines in $\mathcal{H}_B$, $\mathcal{H}_C$ of the three intersections of ℓ_X with the sidelines of triangle ABC.

Two other conics γ_B, γ_C are defined similarly and have two known points in common namely ∞_X and P_a. They must have two other common points lying on a line λ_A which contains X. These two lines are actually homologous under the involution Φ described below.

When ℓ_X rotates about X, λ_A and γ_A meet at two points E, F lying on $\Sigma(X)$ and collinear with X. See figure 10.

Naturally, if λ_A and γ_A are replaced by their homologous λ_B and γ_B or λ_C and γ_C, we obtain the same points E, F.

When ℓ_X is parallel to $\mathcal{O}(X)$ the homologous line λ_A is in fact the (focal) tangent at X to $\Sigma(X)$. It follows that one of the points E, F coincide with X and the other is the intersection of $\Sigma(X)$ with its real asymptote $\mathcal{A}(X)$ which gives its construction since it is parallel to $\mathcal{O}(X)$. From this, the construction of $\mathcal{O}(X)$ is easy since it is the homothetic of this asymptote under $h(X, 1/2)$. Recall that the bisectors of these special cases of ℓ_X, λ_A form the polar conic $\mathcal{C}(\infty_X)$ of ∞_X with respect to $\mathcal{K}(X)$. According to known properties of focal cubics, $\mathcal{C}(\infty_X)$ meets $\Sigma(X)$ at four finite points which are the centers of anallagmaty of $\Sigma(X)$. See figure 11.

The construction of the circle $\mathcal{P}(X)$ which is the polar conic of X in $\Sigma(X)$ is obtained with two lines λ_A intersecting $\Sigma(X)$ at E_1, F_1 and E_2, F_2 as above. It is sufficient to draw the circle passing through X and its harmonic conjugates with respect to E_1, F_1 and E_2, F_2. The focal tangent at X can also be used.

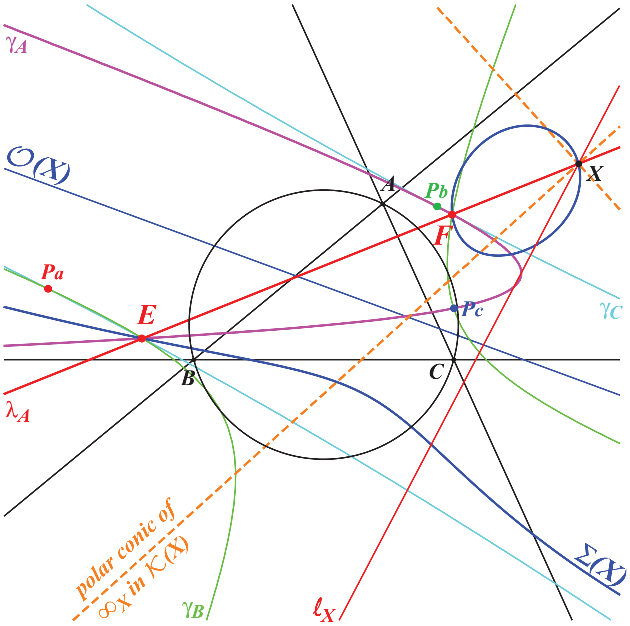

Figure 10: Construction of $\Sigma(X)$ (part 1)

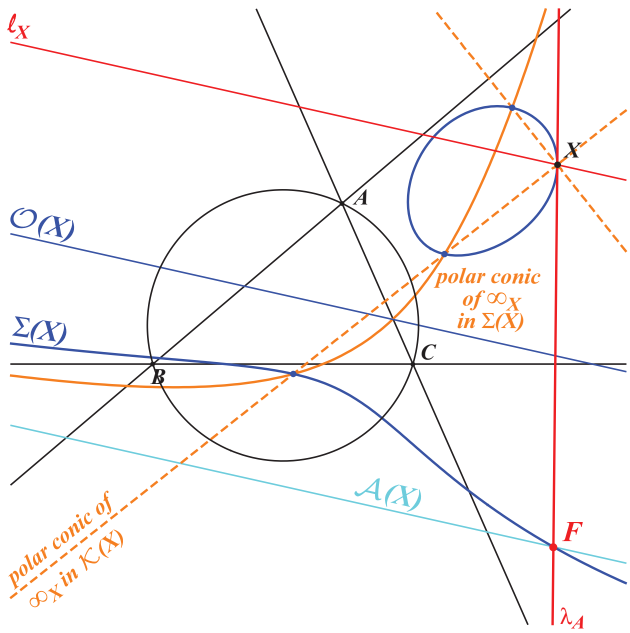

Figure 11: Construction of $\Sigma(X)$ (part 2)

3.4 The related involution Φ

Consider the transformation Φ that maps a point M in the plane onto the center M' of the polar conic of M in $\mathcal{K}(X)$. When M lies on $\Sigma(X)$, this point M' also lies on $\Sigma(X)$ hence Φ globally fixes $\Sigma(X)$ and then, as said above, these points M, M' are conjugate points on $\Sigma(X)$.

We simply recall several properties already mentioned above and elsewhere. See [1] or [5] for example.

Lemma 1 *Φ has one singular point namely X and all the points at infinity are mapped* *onto X.*

Lemma 2 *Φ has four fixed points namely the poles of the line at infinity in $\mathcal{K}(X)$ but only two Φ_1, Φ_2 are real.*

Indeed, the polar conic of any infinite point is a rectangular hyperbola with center X and therefore all these hyperbolas are concentric. In particular, the polar conic $\mathcal{C}(\infty_X)$ of ∞_X (which splits into two perpendicular lines as seen above in §3.2) contains these four points.

Lemma 3 *The two lines composing $\mathcal{C}(\infty_X)$ are the fixes rays of the involution which swaps the lines ℓ_X and λ_A.*

Proposition 4 *Φ is the commutative product of a reflection about one of these lines and an inversion with respect to the circle with diameter $\Phi_1\,\Phi_2$.*

4 Special cubics $\mathcal{K}(X)$

In this section, we study the condition(s) on X such that $\mathcal{K}(X)$ be a certain kind of cubic. See figure 23 at the end of this section and also the table in §8.

4.1 Central cubics $\mathcal{K}(X)$

$\mathcal{K}(X)$ is a central cubic when it contains its radial center X which is also the center of the cubic. Since X always lies on the Hessian $\Sigma(X)$, the polar conic of X splits into the line at infinity and the inflexional tangent at X.

Proposition 5 *$\mathcal{K}(X)$ is a central cubic if and only if X (center and radial center) lies on the first Musselman cubic K026.*

K026 is in fact the central stelloid $\mathcal{K}(X_5)$ passing through X_3, X_4, $E_{626} = X_4X_{51} \cap X_{800}X_{1990}$ (which is the tangential of X_4) and its reflection about X_5.

The Hessian $\Sigma(X)$ of $\mathcal{K}(X)$ is also a central cubic and its center (and singular focus) is that of $\mathcal{K}(X)$. See figure 12 where $X = O$ and $\mathcal{K}(X) =$ K080.

4.2 Axial cubics $\mathcal{K}(X)$

$\mathcal{K}(X)$ is an axial cubic if and only if it is symmetric with respect to a certain (finite) line, the axis $\mathcal{A}$ of the symmetry, which must contain the radial center X of the stelloid $\mathcal{K}(X)$. The point at infinity of $\mathcal{A}$ must be a real point of inflexion on $\mathcal{K}(X)$ hence it must lie on the Hessian $\Sigma(X)$.

From this, two situations can occur since $\Sigma(X)$ has only one real infinite point unless it decomposes into the line at infinity and a circle which is in general not the case.

1. $\Sigma(X)$ contains one and only one infinite point of the McCay cubic and then $\mathcal{K}(X)$ is an axial cubic with one axis of symmetry which is the perpendicular at X to the corresponding asymptote of the McCay cubic.

2. $\Sigma(X)$ contains two (and therefore three) infinite points of the McCay cubic and then $\mathcal{K}(X)$ is an axial cubic with three axes of symmetry which are the perpendiculars at X to the asymptotes of the McCay cubic.

Now, we have to characterize for which points X this can occur and we shall need several lemmas which are obtained after some easy computations.

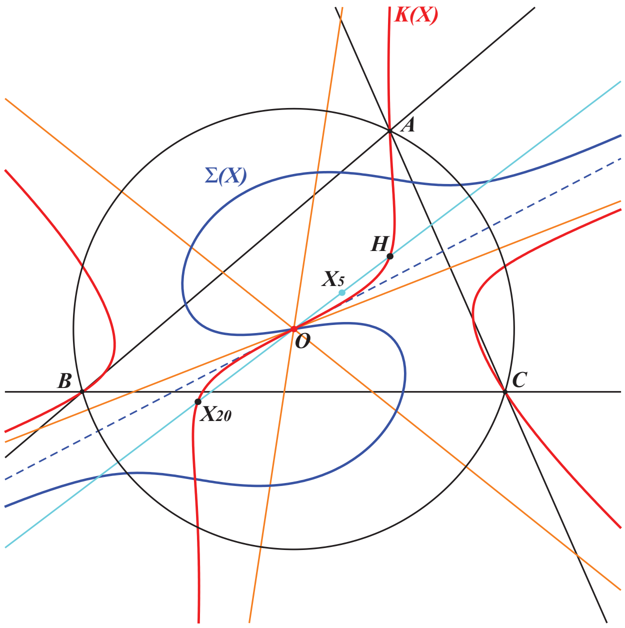

Figure 12: Central cubics $\mathcal{K}(X)$ and $\Sigma(X)$

Lemma 4 *Let P be a point at infinity. The locus of X such that $\Sigma(X)$ contains P is a rectangular hyperbola $\mathcal{H}(P)$.*

$\mathcal{H}(P)$ has its center on the nine-point circle and this center is the complement of the isogonal conjugate of P. The asymptotes of $\mathcal{H}(P)$ are parallel to those of the isogonal transform of the line OP.

Lemma 5 *When P traverses the line at infinity, the rectangular hyperbolas $\mathcal{H}(P)$ belong to a same pencil $\mathcal{F}$ hence they must contain four fixed points $F_i, i \in \{1, 2, 3, 4\}$.*

It is enough to identify at least two of these hyperbolas to obtain the four fixed points.

Let H_A, G_A be the projections of H, G on the sideline BC and let O_A be the antipode of H_A on the nine-point circle.

Lemma 6 *The rectangular hyperbola Γ_A with center O_A, passing through G_A and whose asymptotes are parallel and perpendicular at O_A to the sideline BC is a member of the pencil $\mathcal{F}$.*

In other words, these asymptotes are the reflections about X_5 of the lines BC and AH.

Naturally, two other members Γ_B, Γ_C are defined likewise. Note that Γ_A is the locus of X such that $\Sigma(X)$ contains the infinite point of the line OA hence $\Gamma_A = \mathcal{H}(\infty_{OA})$. See figure 13.

Lemma 7 *The pencil $\mathcal{F}$ contains three other rectangular hyperbolas Γ_1, Γ_2, Γ_3 each passing through one (and therefore only one) of the infinite points of the McCay cubic and obviously the four fixed points F_i of $\mathcal{F}$.*

The centers O_1, O_2, O_3 of Γ_1, Γ_2, Γ_3 are the anticomplements of the vertices N_1, N_2, N_3 of the circumnormal triangle, these latter points lying on the McCay cubic.

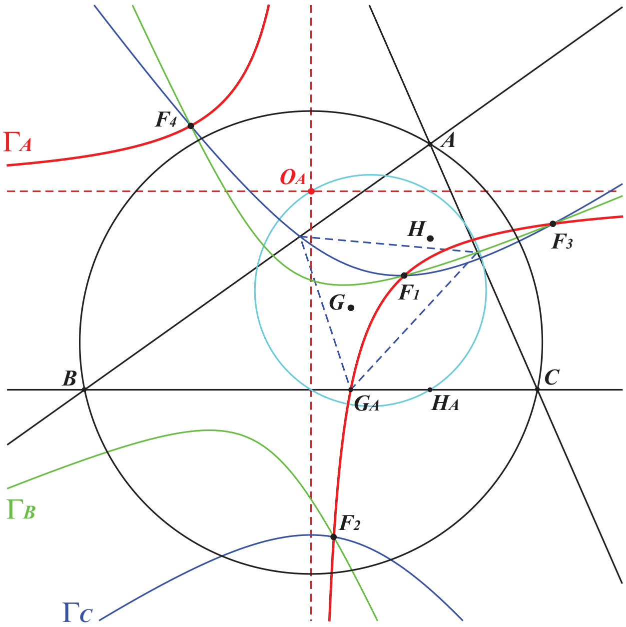

Figure 13: The rectangular hyperbolas Γ_A, Γ_B, Γ_C

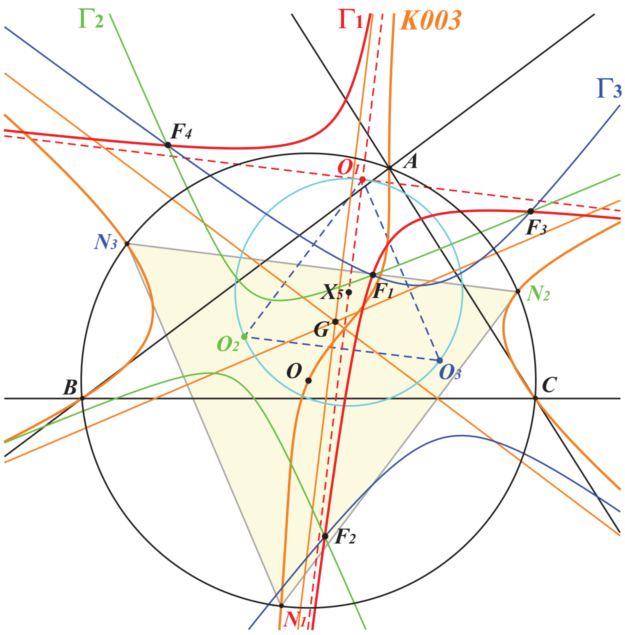

Figure 14: The rectangular hyperbolas Γ_1, Γ_2, Γ_3

Thus, the asymptotes of Γ_1 are the complements of the line ON_1 and its perpendicular at N_1. See figure 14.

Coming back to the two situations mentioned above, we obtain the following results.

Proposition 6 *For any point X lying on one of the hyperbolas Γ_1, Γ_2, Γ_3 and distinct of the four fixed points F_i, the cubic $\mathcal{K}(X)$ is an axial cubic whose axis of symmetry is the perpendicular at X to the corresponding asymptote of the McCay cubic.*

Note that the corresponding Hessian cubic $\Sigma(X)$ must also be an axial cubic with the same axis of symmetry. See figure 15 where the polar conic of X in $\Sigma(X)$ and the orthic line of $\Sigma(X)$ are also represented.

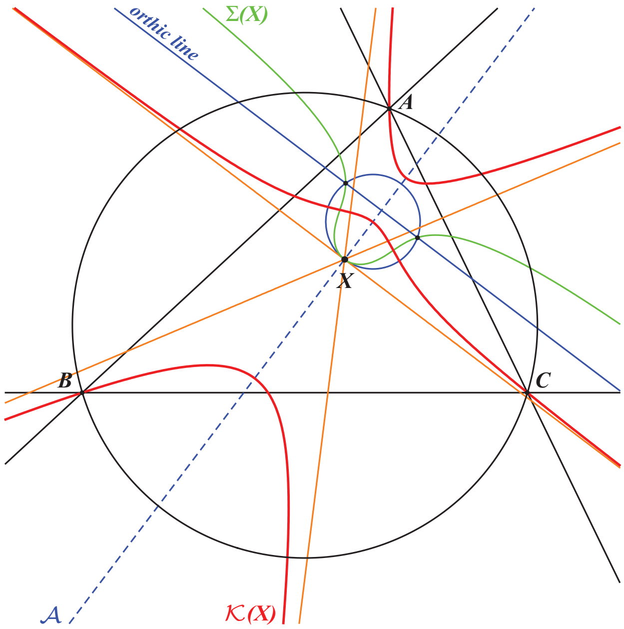

Figure 15: $\mathcal{K}(X)$ with one axis of symmetry

Proposition 7 *If X is one of the four fixed points F_i, the cubic $\mathcal{K}(X)$ is an axial cubic with three axes of symmetry which are the perpendiculars at X to the asymptotes of the McCay cubic.*

Any such cubic $\mathcal{K}(X)$ has three real points of inflexion on the line at infinity which obviously must also lie on the Hessian $\Sigma(X)$. Since $\Sigma(X)$ is circular, it must split into the line at infinity and a conic which is in fact the circle with center X and radius 0. See figure 16.

Additional properties of the points F_i

The pencil $\mathcal{F}$ above also contains three other rectangular hyperbolas γ_A, γ_B, γ_C, each passing through one vertex of the triangle ABC. It follows that the points F_i lie on any circum-cubic of the net generated by the three decomposed cubics which are the union of one of the rectangular hyperbolas γ_x above and the corresponding sideline of ABC i.e. any cubic of the form

$$p\,x\,\gamma_A + q\,y\,\gamma_B + r\,z\,\gamma_C,$$

where $P = p : q : r$ is any point of the plane.

In particular, when $P = G$, we obtain the stelloid K412 which is the locus of the points of concurrence of the asymptotes of all $p\mathcal{K}^+$ with pivot the orthocenter H.

The nine-point circle meets the rectangular hyperbola passing through X_2, X_4, X_6, X_{51}, X_{125}, X_{1209}, X_{1640}, X_{1853}, X_{2574}, X_{2575} at X_{125} and three other points forming a triangle $A'B'C'$ whose in/excenters are precisely the points F_i. This gives a fairly easy conic construction of these points. Note that K412 is actually the McCay cubic for this triangle. The circumcenter of $A'B'C'$ is obviously X_5 and its orthocenter is X_{51}. In other words, this triangle $A'B'C'$ and the orthic triangle have the same Euler line.

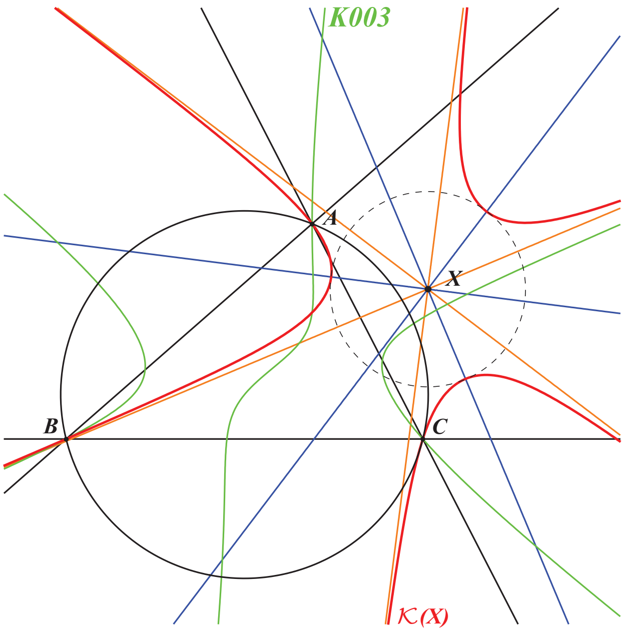

Figure 16: $\mathcal{K}(X)$ with three axes of symmetry

Note that the nine-point circle and the rectangular hyperbola meet again at three points which are the vertices of an equilateral triangle inscribed in K412 which is actually the circum-normal triangle of $A'B'C'$. See figure 17.

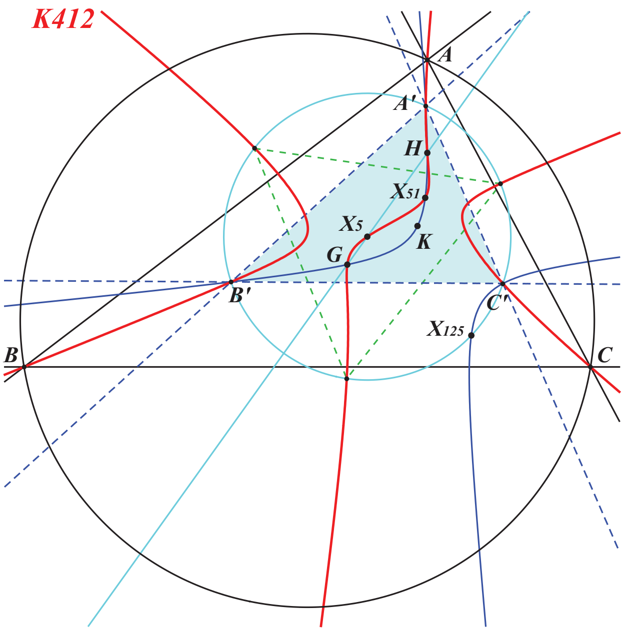

Figure 17: K412 and the triangle $A'B'C'$

We have the following special cases of cubics passing through these four points F_i :

1. With $P = X_6, X_{393}, X_{1249}$ we find three pivotal cubics.
2. More generally, when P lies on K675 $= p\mathcal{K}(X_4, X_4 \times X_{1975})$, we find pseudo-pivotal cubics. This cubic contains X_6, X_{69}, X_{193}, X_{253}, X_{264}, X_{393}, X_{1249}, X_{2998}, X_{3164},

X_{3186}. This gives as many $ps\mathcal{K}$s with

– pseudo-pivot on K677 $= p\mathcal{K}(X_4, X_4 \times X_{183})$ passing through X_2, X_4, X_6, X_{262}, X_{264}, X_{458}, X_{1007},

– pseudo-pole on $p\mathcal{K}(X_5 \times X_{53}, X_5 \times X_{458})$ passing through the barycentric products of X_5 and all the points on the previous cubic.

3. When P lies on the cubic K676 $= n\mathcal{K}(X_4, X_{458}, X_{98})$, we find non-pivotal cubics. This is the case for $P = X_{98}, X_{297}, X_{523}, X_{648}, X_{1113}, X_{1114}, X_{2592}, X_{2593}$ but these cubics are often complicated.

4.3 Cubics $\mathcal{K}(X)$ which are $\mathcal{K}_0$

$\mathcal{K}(X)$ is a $\mathcal{K}_0$ if and only if its equation has no term in xyz. These cubics are sometimes called "apolar cubics" : any two vertices of triangle ABC are conjugated with respect to the polar conic of the remaining vertex.

Proposition 8 *$\mathcal{K}(X)$ is a $\mathcal{K}_0$ if and only if X lies on the parallel (L) at G to the Brocard axis.*

This line contains X_2, X_{51}, X_{262}, X_{263}, X_{373}, X_{511}, X_{2979}, X_{3060} hence the cubic contains Y on the Brocard axis.

All these cubics belong to a same pencil generated by the McCay cubic K003 (obtained when $X = X_2$ and $Y = X_3$) and the decomposed cubic which is the union of the line at infinity and the Kiepert hyperbola (obtained when $X = X_{511}$). With $X = X_{51}$ and $Y = X_{52}$, we find the McCay orthic cubic K049.

4.4 Cubics $\mathcal{K}(X)$ which are $ps\mathcal{K}$

$\mathcal{K}(X)$ is a $ps\mathcal{K}$ (see [4] for further explanations and properties) if and only if its tangents at A, B, C are concurrent or, equivalently, it meets the sidelines of ABC again at three points U, V, W such that ABC and UVW are perspective.

Proposition 9 *$\mathcal{K}(X)$ is a $ps\mathcal{K}$ if and only if X lies on a central cubic (K) with center X_5 which is the image of K044 under $h(X_5, 1/3)$. In this case, the pseudo-pivot lies on K045 and the pseudo-isopivot lies on K350.*

K044 $= p\mathcal{K}(X_{216}, X_4)$ is the Darboux cubic of the orthic triangle. It contains X_3, X_4, X_5, X_{52}, X_{68}, X_{155}. (K) contains X_2, X_5, X_{51}, X_{381} and the homothetic images of X_{68}, X_{155} which are unlisted in the current edition of [9]. The corresponding stelloids are $\mathcal{K}(X_2) =$ K003, $\mathcal{K}(X_5) =$ K026, $\mathcal{K}(X_{51}) =$ K049, $\mathcal{K}(X_{381}) =$ K028.

With X_{68} on K044, we obtain K670 $= ps\mathcal{K}(X_4 \times X_{2165}, X_{5392}, X_4)$ passing through X_4, X_{26}, X_{64}, X_{68}, X_{847} whose radial center is at one third from X_5 to X_{68}. See figure 18.

Cubics $\mathcal{K}(X)$ which are central $ps\mathcal{K}$

Combining the results above and those of §4.1 we find that X must be one of the nine common points of the cubics K026 and (K), one of them being X_5 (giving the cubic K026 itself) and the remaining eight points being always real and two by two symmetric about X_5 and also two by two on a parallel to one of the asymptotes of the McCay cubic. See figure 19.

Proposition 10 *There are exactly nine cubics $\mathcal{K}(X)$ which are central $ps\mathcal{K}$ and one of them is K026.*

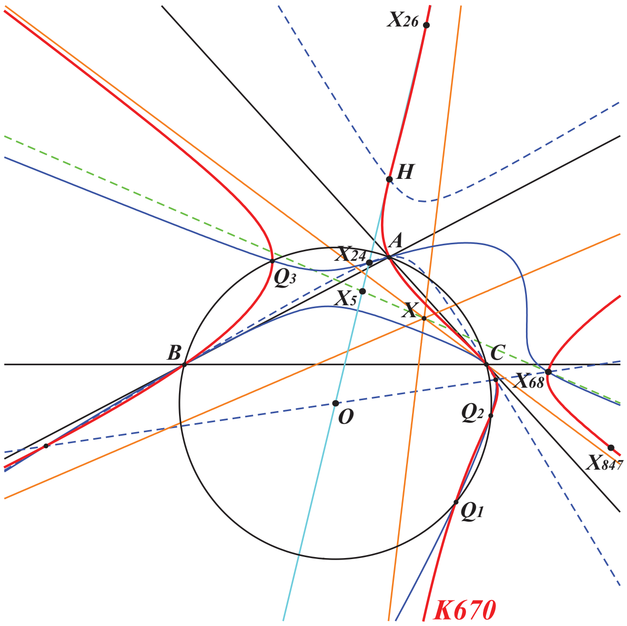

Figure 18: K670, a stelloidal $ps\mathcal{K}$

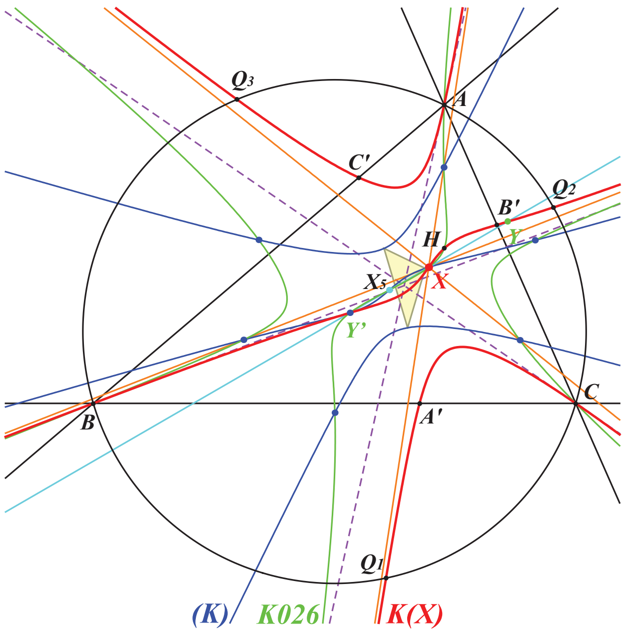

Figure 19: One of the eight central $ps\mathcal{K}$ apart K026

4.5 Cubics $\mathcal{K}(X)$ which are $p\mathcal{K}$

$\mathcal{K}(X)$ is a pivotal cubic $p\mathcal{K}$ if and only if it is simultaneously a $\mathcal{K}_0$ and a $ps\mathcal{K}$.

Proposition 11 *$\mathcal{K}(X)$ is a $p\mathcal{K}$ if and only if X is either X_2 or X_{51} giving the McCay cubic* K003 *itself and the McCay orthic cubic* K049 *respectively.*

4.6 Cubics $\mathcal{K}(X)$ which are $n\mathcal{K}$

$\mathcal{K}(X)$ is a non-pivotal cubic $n\mathcal{K}$ if and only if the points U, V, W above are collinear.

Proposition 12 *$\mathcal{K}(X)$ is a $n\mathcal{K}$ if and only if X lies on a conic (C) which is the image of the circum-conic with center X_5 (and perspector X_{216}) under $h(X_5, 1/3)$.*

This conic is an ellipse if and only if ABC is an acute triangle.

The pole of such cubic $n\mathcal{K}$ lies on the circum-conic with perspector X_{51} passing through X_{112}, X_{1625}, X_{1987}, X_{1989}. The root lies on the unicursal cubic $c\mathcal{K}(\#X_2, X_{264})$ with node G. See [1], chapter 8.

For example, K613 is the cubic with pole X_{112}. It contains X_4, X_{110}, X_{1113}, X_{1114}.

The cubic K669 with pole X_{1989} is also remarkable. Its radial center is at one third from X_5 to X_{265}. It meets the circumcircle at X_{74} and two other points on the perpendicular bisector of OH, these three points on the cubic K668 $= p\mathcal{K}(X_6, X_{265})$. See figure 20.

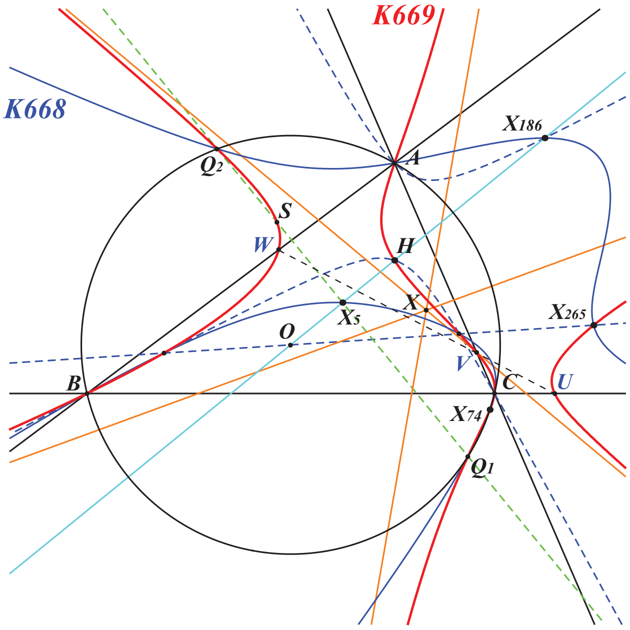

Figure 20: K669, a stelloidal $n\mathcal{K}$

Cubics $\mathcal{K}(X)$ which are central $n\mathcal{K}$

Here again we combine the results above and those of §4.1 and we find that X must be one of the six common points of the cubic K026 and the conic (C), these points being all real and two by two symmetric about X_5.

Indeed, we can parametrize the conic (C) as follows : let $X_{216} = p : q : r$ and $f : g : h = 1 + t : t : -1 - 2t$ at infinity. The point $P_1 = p/f : q/g : r/h$ clearly lies on the circum-conic with center X_5 hence its image P_2 under $h(X_5, 1/3)$ lies on (C).

When the coordinates of P_2 are plugged into the equation of K026, we obtain a polynomial in t of degree 6 which factorizes into two polynomials in t of degree 3 whose discriminants are always positive.

One of these polynomials expresses that the infinite point of the trilinear polar ℓ of the root of $\mathcal{K}(X)$ lies on $\mathcal{K}(X)$ therefore $\mathcal{K}(X)$ must split into ℓ and a circum-conic.

The other expresses that $f : g : h$ is one of the infinite points of the McCay cubic. We can conclude by

Proposition 13 *There are exactly three cubics $\mathcal{K}(X)$ which are central $n\mathcal{K}$. They are obtained when X is the image under $h(X_5, 1/3)$ of the $X_{216}-$isoconjugate of one of the infinite points of the McCay cubic.*

Figure 21 shows one of these three cubics which contains its center X, Y, its reflection Y' in X, H, its reflection H' in X, the traces U, V, W of the trilinear polar of its root.

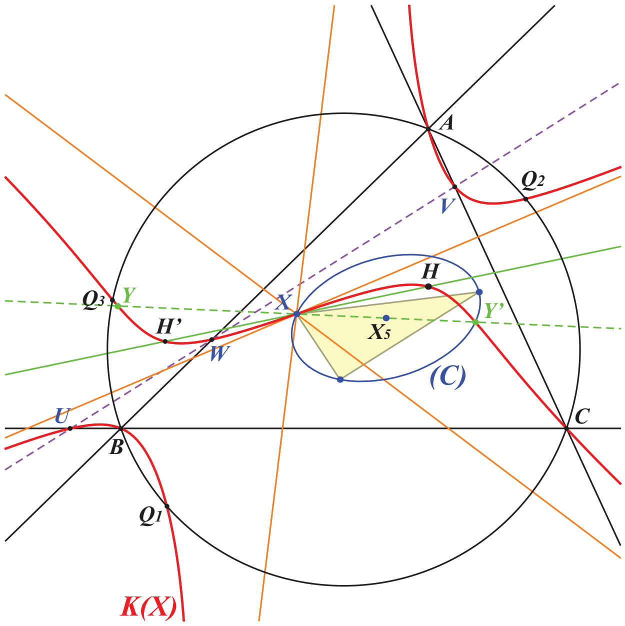

Figure 21: One of the three central $n\mathcal{K}$

4.7 Cubics $\mathcal{K}(X)$ which are nodal

We already know a special nodal cubic namely K028 with node H and radial center X_{381}.

In §2.5, we have seen that the cubics $\mathcal{K}(X)$ passing through a given point $P \neq H$ form a pencil since they already have eight identified common points namely the vertices of ABC, the infinite points of the McCay cubic, the orthocenter H and P. Hence these cubics must have a nineth common point Q which lies on the rectangular circum-hyperbola passing through P and on the line L_P passing through P and the isogonal conjugate P' of P with respect to the pedal triangle of P. This latter line is perfectly defined when $P \neq H$ assuming that P is a finite point different of A, B, C.

When Q coincide with P, the cubic is a nodal cubic with node P.

It is sufficient to express that the polar conic of any point $l : m : n$ in the plane contains P. This gives three conditions on the coefficients of l, m, n of this polar conic and the elimination of the coordinates u, v, w of X gives

Proposition 14 *There is a nodal cubic $\mathcal{K}(X)$ passing through a given point $P \neq H$ and having its node at P if P lies on the antigonal circular quintic* Q038.

Q038 passes through X_1, X_4, X_5, X_{80}, X_{1113}, X_{1114}, X_{1263}, X_{2009}, X_{2010}. It is the locus of P such that P, X_5 and the antigonal of P are collinear.

The construction of Q038 is very easy : it is sufficient to intersect the hyperbola L^*_O of §2.4 with the line passing through X_5 and the center of L^*_O which is the midpoint of H and the fourth point of L^*_O on the circumcircle. See [3], related curves, for other properties of Q038.

When $P = X_1$ and $P =$ midpoint of X_5X_{51}, the nodal cubics are K594 and K054 respectively.

This condition above is necessary but not sufficient. Indeed, P on Q038 can be a node or a flex on the cubic. Let us parametrize the line PQ is with the point $Y_t = (1-t)P + tQ$ and consider the cubic $\mathcal{K}(X_t)$ where X_t is the homothetic of Y_t under $h(X_5, 1/3)$. The polar conic of P degenerates if and only if t is solution of a third degree equation which has a simple solution s and a double solution d.

– $\mathcal{K}(X_d)$ has a flex at P and the polar conic of P splits into the inflexional tangent and the harmonic polar line.

– $\mathcal{K}(X_s)$ has a node at P and the polar conic of P splits into the two nodal tangents at P.

These are represented in figure 22 when $P = X_{80}$.

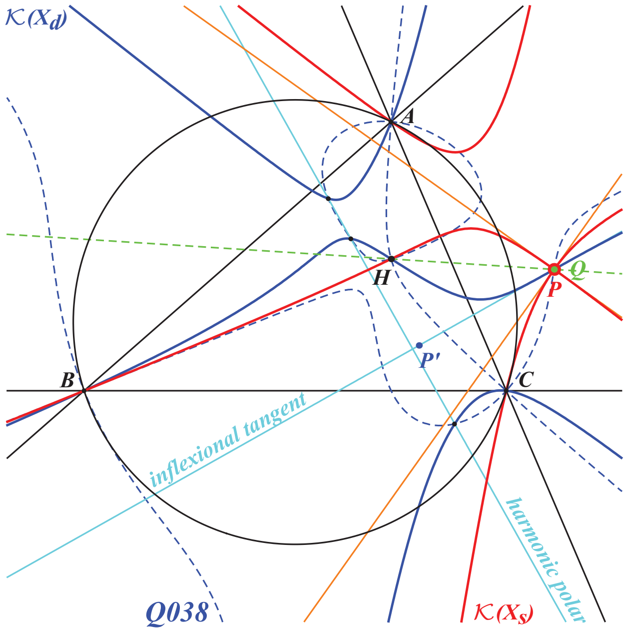

Figure 22: Nodal cubic $\mathcal{K}(X)$

4.8 An overview figure

The following figure 23 gathers together the different loci mentioned above.

5 Groups of pivots of $\mathcal{K}(X)$

Recall that any stelloid can be defined by infinitely many groups of three pivots P_1, P_2, P_3 which are three points suitably chosen on the curve such that the sum of the line angles $\angle(P_1M, \ell) + \angle(P_2M, \ell) + \angle(P_3M, \ell)$ is constant (mod. π) for any point M on the curve and any arbitrary line ℓ. These pivots are not necessarily all distinct.

When one of these pivots is known, say P_1, the remaining pivots are entirely and uniquely determined. Indeed, following Fouret [2], if the isotropic lines passing through P_1 meet the stelloid again at A_1, B_1 and A_2, B_2 respectively then the remaining pivots are $P_2 = A_1J_2 \cap A_2J_1$ and $P_3 = B_1J_2 \cap B_2J_1$ where J_1, J_2 are the circular points at infinity. See [1], chapter 5, for further details.

Since any cubic $\mathcal{K}(X)$ contains H, we can identify the two other corresponding pivots which are the (real) foci F_1, F_2 of the inscribed conic $\gamma(X)$ whose center is the midpoint H' of OY. Note that the radial center X of the stelloid is clearly the centroid of the triangle HF_1F_2. This is in fact true for any group of three pivots. Furthermore,

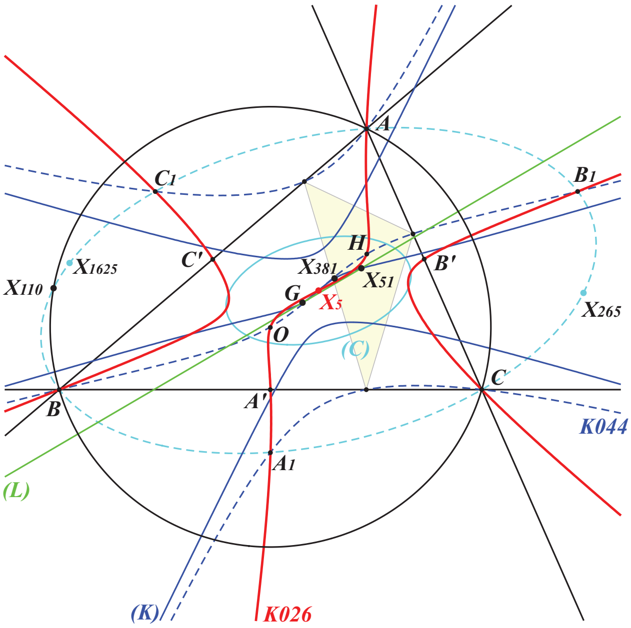

Figure 23: Loci related with special cubics $\mathcal{K}(X)$

the circumcircle of any group of pivots meets $\mathcal{K}(X)$ again at three points which are the vertices of an equilateral triangle. See figure 24.

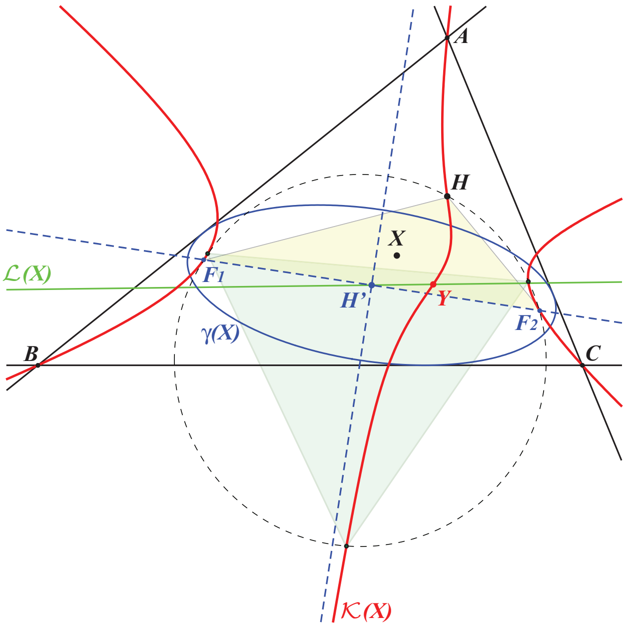

Figure 24: $\mathcal{K}(X)$ and its pivots H, F_1, F_2

Recall that if M, N are any two other points on $\mathcal{K}(X)$, we always have

$$\angle(HM, HN) + \angle(F_1M, F_1N) + \angle(F_2M, F_2N) = 0 \text{ (mod. } \pi).$$

For example, with $X = X_{3545}$, the harmonic conjugate of X_{376} with respect to G and H, we obtain the cubic K358 for which $H' = G$. Hence the two pivots associated with H are the foci of the inscribed Steiner ellipse.

A group of pivots of K582 is G, H, K. For K516 we find H and the Brocard points.

6 Pencils of cubics containing $\mathcal{K}(X)$

6.1 Pencil of cubics generated by the McCay cubic K003 and $\mathcal{K}(X)$

Let $X \neq G$ be a point and denote by $\mathcal{F}_X$ the pencil of cubics generated by the McCay cubic $\mathcal{M} =$ K003 and $\mathcal{K}(X)$. Any cubic of $\mathcal{F}_X$ is a stelloid and contains A, B, C, H, the infinite points of $\mathcal{M}$ and the two isogonal conjugate points Y_1, Y_2 of $\mathcal{M}$ on the line $\mathcal{L}(X) = OY$ and on its isogonal transform $\mathcal{H}(X)$. See figure 1.

A cubic of $\mathcal{F}_X$ can be written

$$\mathcal{K}_\lambda(X) = (1-\lambda)(u+v+w)\,\mathcal{M} + \lambda\,\mathcal{K}(X),$$

where λ is any real number or ∞, in which case its radial center X_λ is the homothetic of X under $h(G, \lambda)$.

By equation (1) we also have

$$\mathcal{K}_\lambda(X) = (u+v+w)\,\mathcal{M} - \lambda\,(x+y+z)\,\mathcal{H}(X),$$

and by equation (2) we obtain

$$\mathcal{K}_\lambda(X) = p\mathcal{K}(X_6, Y_\lambda) + \lambda\,(a^2yz + b^2zx + c^2xy)\,\mathcal{L}(X),$$

where Y_λ is the homothetic of X_λ under $h(X_5, 3)$.

6.2 Pencil of cubics generated by K026 and $\mathcal{K}(X)$

Let $X \neq X_5$. If the point Y defined in §2.1 is replaced by $Y_k = h_k(X)$, where $h_k = h(X_5, 3k)$, then we obtain another McCay stelloid $K_k(X)$ with radial center X_k, the homothetic of Y_k under $h(X_5, 1/3)$.

When k varies, these cubics $K_k(X)$ form another pencil of McCay stelloids which pass through A, B, C, H, the infinite points of $\mathcal{M}$ and the two isogonal conjugate points Z_1, Z_2 defined in §2.4.

Obviously $K_0(X) =$ K026, $K_1(X) = K(X)$ and $K_\infty(X)$ splits into the line at infinity and the rectangular circum-hyperbola passing through Z_1, Z_2. This is the isogonal transform of the parallel at O to the line X_5X.

7 Intersection of $\mathcal{K}(X)$ with triads of lines parallel to the asymptotes

Let $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_3$ be three lines, each being parallel to one asymptote of $\mathcal{K}(X)$ or K003.

7.1 Concurring lines

When $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_3$ concur at $P \neq X$, they meet $\mathcal{K}(X)$ again at six (finite) points which lie on a same rectangular hyperbola. See figure 25.

7.2 Lines forming an equilateral triangle with center X.

In this case, the lines $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_3$ meet $\mathcal{K}(X)$ again at six (finite) points which lie on a same circle. See figure 26.

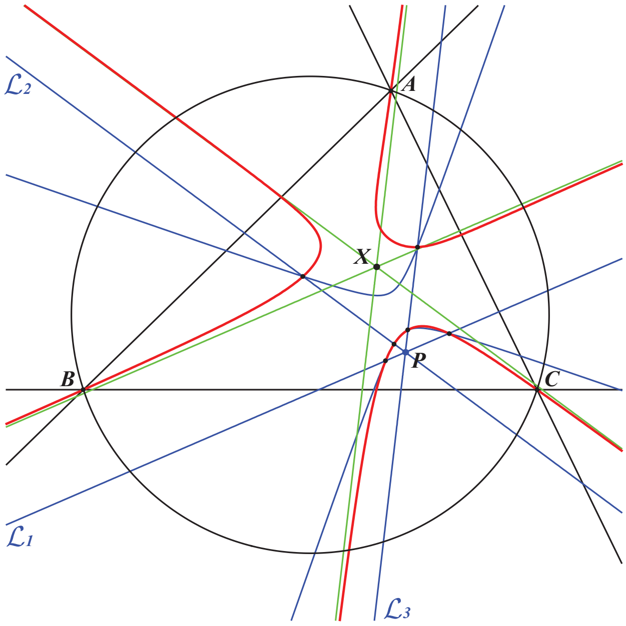

Figure 25: $\mathcal{K}(X)$ and concurring lines parallel to the asymptotes

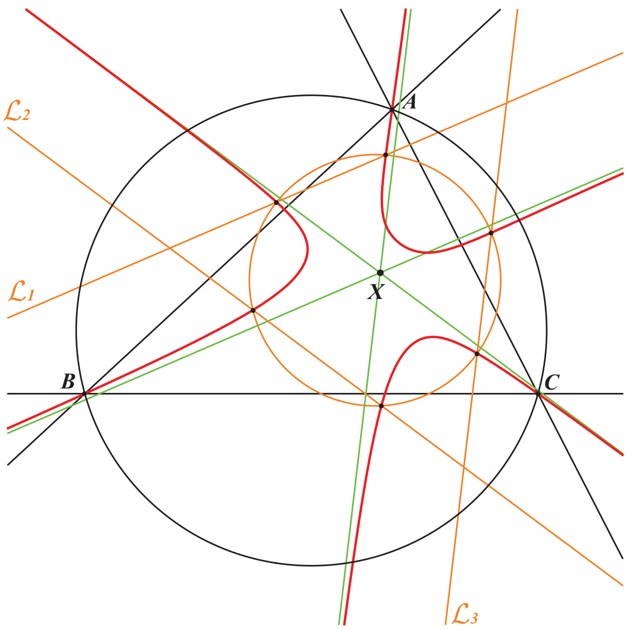

Figure 26: $\mathcal{K}(X)$ and lines parallel to the asymptotes

8 Table of stelloids $\mathcal{K}(X)$

The following table 1 presents a list of all the McCay stelloids already catalogued in [3]. Note that "c" denotes a central cubic, "n" denotes a nodal cubic.

Table 1: A selection of catalogued McCay stelloids

cubic	X	c	n	type	centers on the curve
K003	X_2			$p\mathcal{K}$	X_1, X_3, X_4, X_{1075}, X_{1745}, X_{3362}
K026	X_5	c		$ps\mathcal{K}$	X_3, X_4, X_5
K028	X_{381}		n	$ps\mathcal{K}$	X_3, X_4, X_8, X_{76}, X_{847}
K049	X_{51}			$p\mathcal{K}$	X_4, X_5, X_{52}, X_{847}
K054	midpoint of X_5, X_{51}		n		X_4, X_5, X_{143}
K071	reflection of X_{51} in X_5			$ps\mathcal{K}$	X_4, X_5, X_{20}, X_{76}
K080	X_3	c		$sp\mathcal{K}$	X_3, X_4, X_{20}, X_{1670}, X_{1671}
K115	$X_2X_{511} \cap X_6X_{22}$			$\mathcal{K}_0$	X_4
K268	$X_2X_{511} \cap X_3X_{64}$			$\mathcal{K}_0$	X_4, X_{20}, X_{140}
K309	X_{5054}				X_3, X_4, X_{376}, X_{1340}, X_{1341}
K358	X_{3545}				X_3, X_4, X_{381}
K412	$\overrightarrow{X_5X} = 1/3\ \overrightarrow{X_5X_{51}}$				X_2, X_4, X_5, X_{51}, X_{262}
K514	$X_3X_{1506} \cap X_{20}X_{32}$				X_4, X_{15}, X_{16}, X_{39}
K516	X_{262}			$\mathcal{K}_0$	X_4, X_{3095}
K525	X_4	c			X_3, X_4, X_{382}
K580	X_{568}				X_4, X_{847}
K581	X_{5055}				X_2, X_3, X_4, X_{262}
K582	$\overrightarrow{KX} = 1/3\ \overrightarrow{KX_{381}}$				X_2, X_4, X_6, X_{262}
K594	$\overrightarrow{IX} = 1/3\ \overrightarrow{IH}$		n		X_1, X_4, X_{1482}
K613	$\overrightarrow{X_5X} = 1/3\ \overrightarrow{X_5X_{110}}$			$n\mathcal{K}$	X_4, X_{110}, X_{1113}, X_{1114}
K643	centroid of OHK			$sp\mathcal{K}$	X_4, X_6, X_{4846}
K665	X_{549}				X_3, X_4, X_{39}, X_{550}
K669	$\overrightarrow{X_5X} = 1/3\ \overrightarrow{X_5X_{265}}$			$n\mathcal{K}$	X_4, X_{74}, X_{265}
K670	$\overrightarrow{X_5X} = 1/3\ \overrightarrow{X_5X_{68}}$			$ps\mathcal{K}$	X_4, X_{26}, X_{64}, X_{68}, X_{847}
K708	$\overrightarrow{X_4X} = 2/3\ \overrightarrow{X_4X_6}$			$sp\mathcal{K}$	X_4, X_{1344}, X_{1345}, X_{1351}
K714	$X_3X_{107} \cap X_5X_{53}$			$sp\mathcal{K}$	X_4

References

[1] Ehrmann J.P. and Gibert B., *Special Isocubics in the Triangle Plane*, available at http://bernard.gibert.pagesperso-orange.fr

[2] Fouret G., *Sur quelques propriétés géométriques des stelloïdes*, Comptes rendus hebdomadaires des séances de l'Académie des Sciences, tome 106, (1888), pp. 342–345.

[3] Gibert B., *Cubics in the Triangle Plane*, available at http://bernard.gibert.pagesperso-orange.fr

[4] Gibert B., *Pseudo-Pivotal Cubics and Poristic Triangles*, available at http://bernard.gibert.pagesperso-orange.fr

[5] Gibert B., *Inscribed Cardioids and Eckart Cubics*, available at http://bernard.gibert.pagesperso-orange.fr

[6] Gibert B., *How Pivotal Isocubics Intersect the Circumcircle*

Forum Geometricorum, Volume 7 (2007) 211–229.

[7] Gibert B., *The Lemoine Cubic and its Generalizations*
Forum Geometricorum, Volume 2 (2002) 47–63.

[8] Kimberling C., *Triangle Centers and Central Triangles*, Congressus Numerantium, 129 (1998) 1–295.

[9] Kimberling C., *Encyclopedia of Triangle Centers*, 2000-2014
http://www2.evansville.edu/ck6/encyclopedia
http://faculty.evansville.edu/ck6/encyclopedia/ETC.html