Pseudo-Pivotal Cubics and Poristic Triangles

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Abstract

We try to slightly weaken the characteristics of a pivotal cubic in order to introduce a family of cubics called pseudo-pivotal cubics. We show that these cubics share several properties with the corresponding pivotal cubics related to poristic triangles. We also study special pseudo-pivotal cubics and the pencils or nets they form.

Notations used in this paper :

The points cM, aM, tM, gM are respectively the complement, the anticomplement, the isotomic conjugate, the isogonal conjugate of M.

The symbols $\times$ and $\div$ denote the barycentric product and quotient respectively. $M \times M = M^2$ is the barycentric square of M.

The symbols / and $*$ are used for Ceva conjugation and isoconjugation.

1 Introduction and definitions

1.1 Pivotal cubics $p\mathcal{K}(\Omega, P)$

Let $\Omega = p : q : r$ and $P = u : v : w$ be two points given in barycentric coordinates in the plane of the reference triangle ABC.

The pivotal cubic $p\mathcal{K}(\Omega, P)$ is the locus of point M such that its pivot P, M and the $\Omega-$isoconjugate M^* of M are collinear, hence the cubic $p\mathcal{K}(\Omega, P)$ is invariant under $\Omega-$isoconjugation.

This cubic meets the sidelines of ABC at the vertices P_a, P_b, P_c of the cevian triangle of the pivot P and the tangents at A, B, C to the cubic meet at the isopivot P^*.

The equation of $p\mathcal{K}(\Omega, P)$ is

$$\sum_{\text{cyclic}} x^2(rvy - qwz) = 0 \iff \sum_{\text{cyclic}} p(wy - vz)yz = 0. \tag{1}$$

We briefly recall several properties $p\mathcal{K}(\Omega, P)$:

- The polar conic of P is the diagonal conic passing through P (with tangent the line PP^*) and the vertices of the anticevian triangle of P. This also contains the (real or not) fixed points of the isoconjugation with pole Ω.
- The polar conic of P^* is the circum-conic passing through P and P^*. This has perspector the intersection of the trilinear polars of P and P^* with first coordinate $pu(rv^2 - qw^2)$. The cubic $p\mathcal{K}(\Omega, P)$ is invariant under $P-$Ceva conjugation and any two $P-$Ceva conjugate points M and P/M on the cubic are collinear with P^*.
- The polar conic of P/P^* ($P-$Ceva conjugate of P^*, a point on the cubic) contains P_a, P_b, P_c, P^* and P/P^*. Any two points P/M and P/M^* on the cubic are collinear with $(P/P^*)^*$.

These pivotal cubics $p\mathcal{K}(\Omega, P)$ always contain their pivot P but they can never be unicursal (nodal or cuspidal) cubics unless they degenerate.

For this reason, we plan in this paper to slightly alter the characteristics above in order to obtain a family of cubics able to contain unicursal cubics but, doing this, we wish to keep as many valid properties as possible. Thus, we wish to keep the facts that the tangents at A, B, C be concurrent and that the traces of the cubic on the sidelines of ABC be the vertices of a cevian triangle but we will no longer impose that the cubic necessarily contains the perspector of this cevian triangle and ABC.

1.2 Pseudo-pivotal cubics $ps\mathcal{K}(\Omega, P, \lambda)$ and $ps\mathcal{K}(\Omega, P, M)$

For any real number λ, we define as parent cubic of $p\mathcal{K}(\Omega, P)$ the cubic $ps\mathcal{K}(\Omega, P, \lambda)$ with equation :

$$\sum_{\text{cyclic}} x^2(rvy - qwz) + \lambda\, xyz = 0 \iff \sum_{\text{cyclic}} p(wy - vz)yz + \lambda\, xyz = 0, \tag{2}$$

we shall call pseudo-pivotal cubic since it meets the sidelines of ABC at P_a, P_b, P_c and since it has the same tangents at A, B, C as the corresponding $p\mathcal{K}(\Omega, P)$, these tangents concurring at P^*. For these reasons, the points P, P^*, Ω are called pseudo-pivot, pseudo-isopivot, pseudo-pole of $ps\mathcal{K}(\Omega, P, \lambda)$ although the first two do not lie on the cubic unless it is a $p\mathcal{K}$ obtained when $\lambda = 0$.

It is then clear that $ps\mathcal{K}(\Omega, P, 0) = p\mathcal{K}(\Omega, P)$ and that the $\Omega-$isoconjugate of $ps\mathcal{K}(\Omega, P, \lambda)$ is $ps\mathcal{K}(\Omega, P, -\lambda)$. It is also evident that these cubics $ps\mathcal{K}(\Omega, P, \lambda)$ form a pencil and we shall denote by $ps\mathcal{K}(\Omega, P, M)$ the unique cubic passing through a given point M not lying on the sidelines of ABC. It follows that the $\Omega-$isoconjugate of $ps\mathcal{K}(\Omega, P, M)$ is $ps\mathcal{K}(\Omega, P, M^*)$.

The cubic $ps\mathcal{K}(\Omega, P, M)$ contains P when it is a pivotal cubic and in this case it meets the line MP at M^*. Recall that, in this case, the cubic is invariant under $\Omega-$isoconjugation and $P-$Ceva conjugation and therefore contains P^*, P/M and also P/M^*, $(P/M)^*$.

In [6] we had already met a special case of cubics $ps\mathcal{K}$ denoted by $\mathcal{K}(Q)$ and the equation (13) on page 87 clearly shows that these cubics $\mathcal{K}(Q)$ are actually $ps\mathcal{K}(Q, tgQ, Q)$. These cubics always contain the orthocenter H of ABC and their tangents at A, B, C are the symmedians. They are generally not pivotal cubics unless Q lies on the circumhyperbola passing through G and K in which case the cubic $\mathcal{K}(Q)$ becomes a pivotal cubic with pole Q and pivot tgQ on the Kiepert hyperbola.

If $ps\mathcal{K}(\Omega, P, M)$ does not contain P, it is not a pivotal cubic and it is not invariant under $\Omega-$isoconjugation nor $P-$Ceva conjugation. One remarkable fact is

Proposition 1 *A non-pivotal $ps\mathcal{K}(\Omega, P, M)$ can be transformed into $ps\mathcal{K}(\Omega, P, M^*)$ either by $\Omega-$isoconjugation or $P-$Ceva conjugation.*

which immediatly gives

Proposition 2 *$ps\mathcal{K}(\Omega, P, M)$ is invariant under the product of $\Omega-$isoconjugation and $P-$Ceva conjugation. Furthermore, this invariance is independent of the order used in the transformations.*

and

Proposition 3 *Any cubic $ps\mathcal{K}(\Omega, P, M)$ contains M, $P_1 = P/M^*$ and $P_2 = (P/M)^*$.*

Naturally, these two latter points may be transformed again to give two new points, for example $P_3 = P/P_1^*$ we shall meet in the construction the cubic, and the process repeats as long as we wish.

If $ps\mathcal{K}(\Omega, P, M)$ is not a pivotal cubic then it meets the line MP at two (not necessarily real nor distinct) points M_1, M_2 lying on the circum-conic passing through M^* with tangent at this point passing through P^*. Note that these two points M_1, M_2 are homologous under the isoconjugation that swaps P and M^*.

The conic has a sixth common point with the cubic which is $P_2 = (P/M)^*$. See figure 1 where the cubic is **K565** $= ps\mathcal{K}(X_6, X_2, X_{111})$, a parent of the Thomson cubic **K002**, with two real points $M_1 = X_{111}$, $M_2 = X_{2482}$. $(P/M)^*$ is unlisted in [8].

In the next section, we shall see that these cubics $ps\mathcal{K}(\Omega, P, \lambda)$ share with $p\mathcal{K}(\Omega, P)$ several other common properties related to poristic triangles.

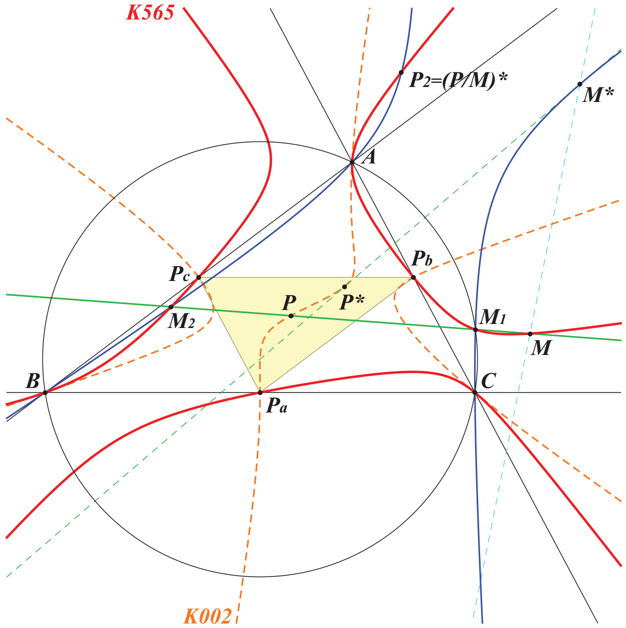

Figure 1: The Thomson cubic **K002** with its parent cubic **K565**

1.3 Construction of $ps\mathcal{K}(\Omega, P, M)$

Let Δ be the trilinear polar of the crossconjugate of M^* and P and let V be a variable point on Δ. The circum-conic Γ_1 passing through P_2 and V meets Δ again at W.

The circum-conic Γ_2 of triangle $P_aP_bP_c$ passing through V and W meets Γ_1 again at two points X, Y on $ps\mathcal{K}(\Omega, P, M)$ collinear with M.

Note that Γ_2 always contains P_3 thus the coresidual of P_a, P_b, P_c, P_3 is M which is also the coresidual of A, B, C, P_2. This means that any conic meeting $ps\mathcal{K}(\Omega, P, M)$ at A, B, C, P_2 (resp. P_a, P_b, P_c, P_3) has two other common points with the cubic collinear with M.

This gives other points on the cubic such as $A' = MP_a \cap AP_2$, $B' = MP_b \cap BP_2$, $C' = MP_c \cap CP_2$.

Remark : Δ is a common chord of two conics γ_1 and γ_2 which also have the line MP_1 as other common chord. γ_1 is the circum-conic of ABC passing through P_1, P_2 and γ_2 is the circum-conic of $P_aP_bP_c$ passing through P_1, P_3. These two conics also contain $Q_1 = MP_1 \cap P_2P_3$, another point on the cubic.

2 Cubics $ps\mathcal{K}$ and poristic triangles

2.1 Generalities

Let $\Gamma(P^*)$ be the circum-conic with perspector P^* and let $\gamma(P)$ be the in-conic with perspector P. $\Gamma(P^*)$ and $\gamma(P)$ are respectively the $\Omega-$isoconjugate and the $P-$Ceva conjugate of the trilinear polar $\mathcal{T}(P)$ of P. Thus, for any point M on $\mathcal{T}(P)$,

– the point M^* lies on $\Gamma(P^*)$ and the tangent at this point passes through the second intersection of $\mathcal{T}(P)$ with the circum-conic through M and M^*,

– the point P/M lies on $\gamma(P)$ and the tangent at this point is the trilinear polar of M.

$\Gamma(P^*)$ meets $ps\mathcal{K}(\Omega, P, \lambda)$ at A, B, C and three other points Q_1, Q_2, Q_3. $\gamma(P)$ meets $ps\mathcal{K}(\Omega, P, \lambda)$ at P_a, P_b, P_c and three other points R_1, R_2, R_3. This gives twelve points on the cubic $ps\mathcal{K}(\Omega, P, \lambda)$ to which we can add P, P^*, P/P^* when $\lambda = 0$ i.e. when the cubic is pivotal. Beware that these six points Q_i, R_i depend of λ.

Figure 2 shows the Grebe cubic **K102** $= p\mathcal{K}(X_6, X_6)$ with two of its parents namely those passing through O and H, one being the isogonal transform of the other. $\Gamma(P^*)$ is the Steiner ellipse and $\gamma(P)$ is the Brocard ellipse.

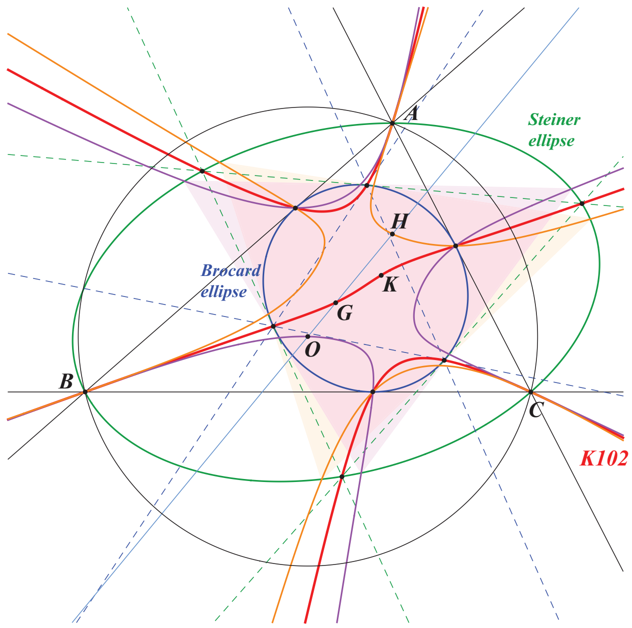

Figure 2: The Grebe cubic **K102** with two of its parent cubics $ps\mathcal{K}$

2.2 Properties and related loci

We wish to examine the properties of these triangles $Q_1Q_2Q_3$ and $R_1R_2R_3$.

Most of the following results are obtained with (sometimes quite heavy) calculations involving symmetric functions of the roots of third degree polynomials.

Recall that they remain valid for all pivotal cubics.

Theorem 1 *For any λ, the triangles $Q_1Q_2Q_3$ and $R_1R_2R_3$ are poristic triangles with respect to $\Gamma(P^*)$ and $\gamma(P)$.*

In other words, $\gamma(P)$ is inscribed in $Q_1Q_2Q_3$ and the contacts of the sidelines of $Q_1Q_2Q_3$ with $\gamma(P)$ are the points R_1, R_2, R_3.

If Q_1 is given on Γ then the line Q_2Q_3 is the trilinear polar of $P \times (P^* \div Q_1)$ and R_1 is $P \times (P^* \div Q_1)^2$. Note that $P^* \div Q_1$ is a point on the line at infinity.

Theorem 2 *The tangents at Q_1, Q_2, Q_3 to $ps\mathcal{K}(\Omega, P, \lambda)$ are concurrent at a point $X(\lambda)$. When λ varies, the locus of $X(\lambda)$ is a conic $\mathcal{C}_X$ passing through P^*.*

$\mathcal{C}_X$ has equation

$$\left(\sum_{\text{cyclic}} p\,v^2w^2\right)\left(\sum_{\text{cyclic}} p\,vw\,yz\right) - 3\,u^2v^2w^2\left(\sum_{\text{cyclic}} qr\,x^2\right) = 0. \tag{3}$$

Note that $\sum_{\text{cyclic}} p\,vw\,yz = 0$ is the equation of $\Gamma(P^*)$. See figure 3.

Theorem 3 *The lines Q_1R_1, Q_2R_2, Q_3R_3 are concurrent at a point $Y(\lambda)$. When λ varies, the locus of $Y(\lambda)$ is a conic $\mathcal{C}_Y$ passing through P.*

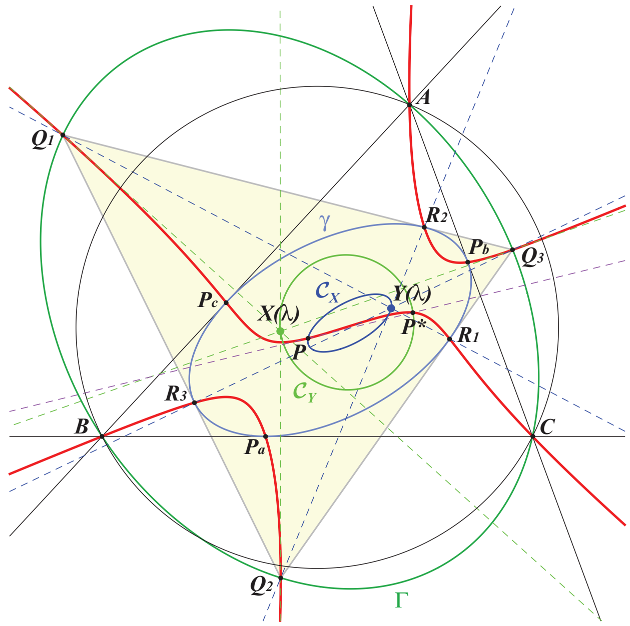

Figure 3: Poristic triangles $Q_1Q_2Q_3$ and $R_1R_2R_3$ with the conics $\mathcal{C}_X$ and $\mathcal{C}_Y$

In other words, the two triangles $Q_1Q_2Q_3$ and $R_1R_2R_3$ are perspective at $Y(\lambda)$.
$\mathcal{C}_Y$ has equation

$$\left(\sum_{\text{cyclic}} p\, v^2w^2\right)\left(\sum_{\text{cyclic}} vw(vw\, x^2 - 2u^2\, yz)\right) + 3\, u^2v^2w^2 \left(\sum_{\text{cyclic}} p\, vw\, yz\right) = 0. \tag{4}$$

Note that $\sum_{\text{cyclic}} vw(vw\, x^2 - 2u^2\, yz) = 0$ is the equation of $\gamma(P)$. This shows that $\mathcal{C}_Y$ is a member of the pencil of conics generated by $\Gamma(P^*)$ and $\gamma(P)$. See figure 3.

Theorem 4 *When λ varies, the triangle $R_1R_2R_3$ circumscribes a fixed conic $\mathcal{C}_Z$ inscribed in the cevian triangle $P_aP_bP_c$ of P.*

$\mathcal{C}_Z$ has equation

$$\sum_{\text{cyclic}} v^2w^2\left[p^2v^2w^2(-vwx + wuy + uvz)^2 - 2qru^4(vwx + wuy - uvz)(vwx - wuy + uvz)\right] = 0. \tag{5}$$

Its center has first coordinate $u(v+w)\left(pv^2w^2 + qu^2w^2 + ru^2v^2\right) + 2p\, u^2v^2w^2$ and clearly lies on the line passing through Ω and the center of $\gamma(P)$. See figure 4.

Theorem 5 *When λ varies, the centroid of $Q_1Q_2Q_3$ lies on a conic $\mathcal{C}_G$ passing through the centroid G of ABC and meeting the line at infinity at the same points as $\Gamma(P^*)$.*

It follows that this conic is a circle if and only if $P^* = X_6$ i.e. if and only if Ω is the isogonal of the isotomic of P. Obviously, $\Gamma(P^*)$ is then the circumcircle $(\mathcal{O})$ of ABC.
Figure 5 shows $p\mathcal{K}(X_3, X_{69})$ (in red) with two of its parent cubics. $\gamma(P)$ and $\mathcal{C}_G$ have both center O.

2.3 Example

One very remarkable example arises when $P = X_{264}$ hence $\Omega = H$. Indeed, this is the only case when $\mathcal{C}_G$ is reduced to the point G. In other words, all the triangles $Q_1Q_2Q_3$ are inscribed in $(\mathcal{O})$ and have the

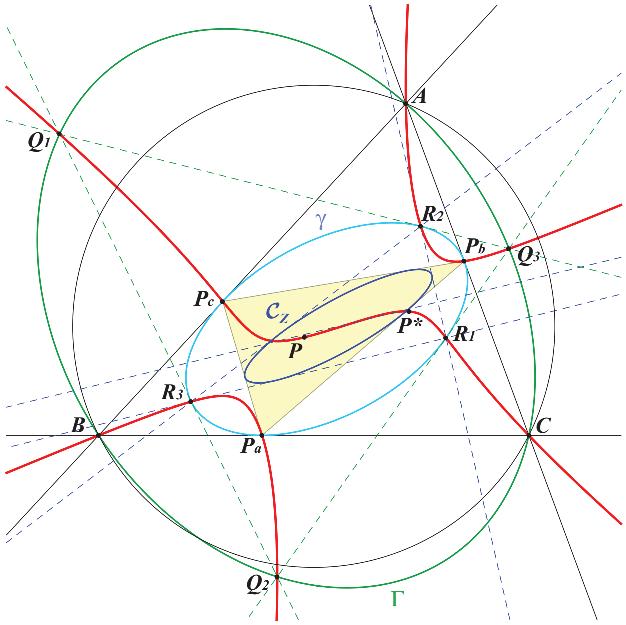

Figure 4: Poristic triangles $Q_1Q_2Q_3$ and $R_1R_2R_3$ with the conic $\mathcal{C}_Z$

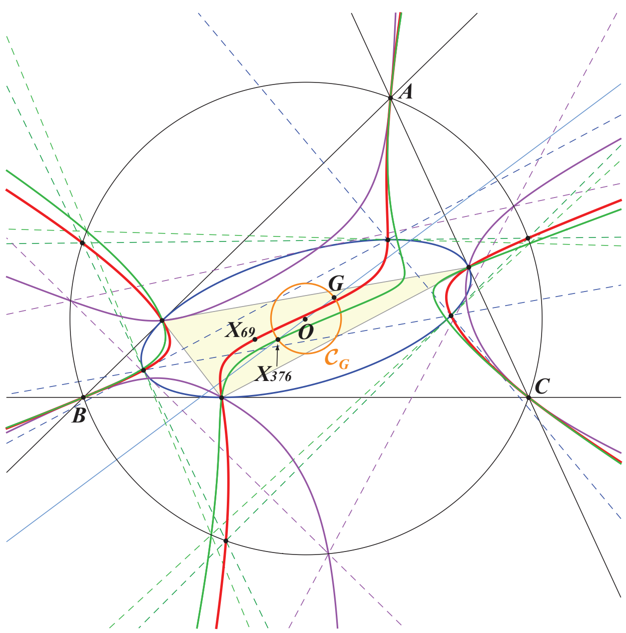

Figure 5: Locus $\mathcal{C}_G$ of centroids of poristic triangles $Q_1Q_2Q_3$

same centroid therefore the same Euler line and also the same orthic axis, that of ABC. The conic $\gamma(P)$ is the McBeath inconic with foci O and H. Recall that the tangents at A, B, C pass through K and those at Q_1, Q_2, Q_3 concur at the Lemoine point of triangle $Q_1Q_2Q_3$. This latter point lies on the circle with diameter $X_{1344}X_{1345}$ passing through X_6.

The corresponding pencil formed by the cubics $ps\mathcal{K}(X_4, X_{264}, \lambda)$ contains several interesting cubics and, in particular, two nodal cubics. This will be generalized in another section below.

The former cubic is **K028** $= ps\mathcal{K}(X_4, X_{264}, X_3)$, the third Musselman cubic with node H, an equilateral cubic with asymptotes parallel to those of the McCay cubic **K003** and concurring at X_{381}, the midpoint of GH.

The latter cubic is **K555** $= ps\mathcal{K}(X_4, X_{264}, X_2)$, the $H-$isoconjugate of **K028**, and is very briefly mentioned in [9] although it has many special properties. It is also the isogonal transform of **K260**. Its node is G with nodal tangents passing through X_{2592} and X_{2593}. This cubic contains X_2, X_{25}, X_{278}, X_{1073}, X_{1993}, X_{2052}, X_{3190}. It meets the orthic axis at three points with tangents concurring on the Euler line at the point with first coordinate $2 + \dfrac{a^2}{2S_A}$. See figure 6.

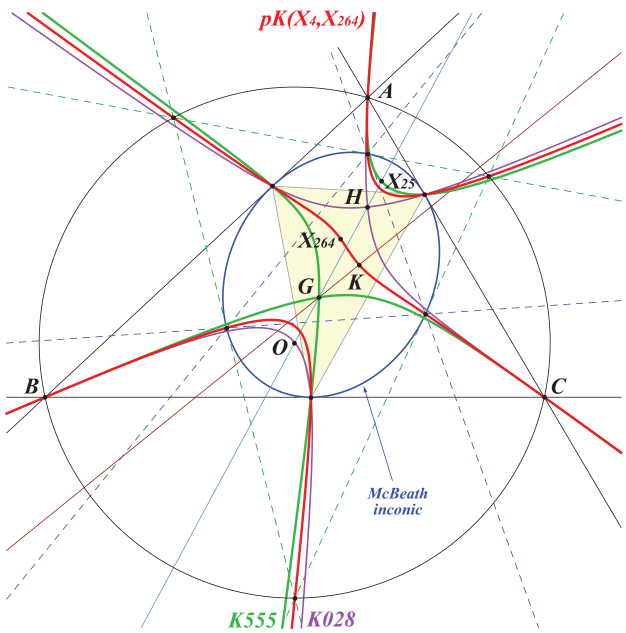

Figure 6: Cubics $ps\mathcal{K}(X_4, X_{264}, \lambda)$ with **K028**

Another interesting cubic of this pencil is **K556** $= ps\mathcal{K}(X_4, X_{264}, X_{523})$ since it is an axial cubic symmetric about the Euler line. See below for further informations about axial cubics.

Theorem 6 *When λ varies, the centroid of $R_1R_2R_3$ lies on a conic $\mathcal{C}'_G$ passing through the centroid of the cevian triangle of P and meeting the line at infinity at the same points as $\gamma(P)$.*

It follows that this conic is a circle if and only if $P = X_7$ (or one of its extraversions) and obviously, $\gamma(P)$ is then the incircle (or an excircle) of ABC.

When we combine the results of theorems 5 and 6 we find that the loci of the centroids of both triangles $Q_1Q_2Q_3$, $R_1R_2R_3$ are simultaneously circles if and only if $P = X_7$ hence $\Omega = X_{56}$ (and similarly with the extraversions) although $\mathcal{C}'_G$ is reduced to the point X_{354}, the centroid of the intouch triangle. The pseudo-isopivot is the Lemoine point K hence $Q_1Q_2Q_3$ is inscribed in the circumcircle and recall that $R_1R_2R_3$ is inscribed in the incircle. These two triangles are the "proper" Poncelet poristic triangles.

This gives an interesting pencil of cubics $ps\mathcal{K}(X_{56}, X_7, \lambda)$ such that the centroid of $R_1R_2R_3$ is X_{354} for any λ. Note that the tangents at A, B, C are the symmedians.

In particular we find **K360** $= ps\mathcal{K}(X_{56}, X_7, X_1)$ (a Lemoine generalized cubic with node X_1), **K577** $= ps\mathcal{K}(X_{56}, X_7, X_2)$ (the $X_{56}-$isoconjugate of **K360** with node X_{56}), **K578** $= ps\mathcal{K}(X_{56}, X_7, X_{513})$ (an axial cubic symmetric about the line OI), the pivotal cubic **K631** $= p\mathcal{K}(X_{56}, X_7)$ passing through X_6, X_7, X_{509}, X_{1486}. See figure 7.

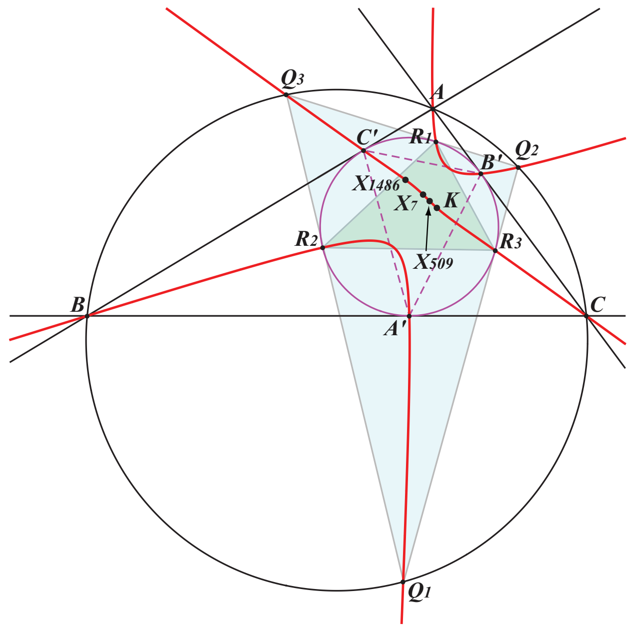

Figure 7: The cubic **K631** $= p\mathcal{K}(X_{56}, X_7)$

2.4 Equilateral poristic triangles

In general, for given points $\Omega = p : q : r$ and $P = u : v : w$, there is no cubic $ps\mathcal{K}(\Omega, P, \lambda)$ such that the triangle $Q_1Q_2Q_3$ is equilateral. Indeed, when we write that the decomposed cubic that consists of the union of its sidelines is an equilateral cubic, we obtain two conditions of degree 2 with respect to λ and these conditions are generally incompatible. The elimination of λ yields to a complicated condition of gobal degree 4 with respect to the coordinates of Ω and 8 with respect to the coordinates of P.

Similarly, the condition for which there exists an equilateral triangle $R_1R_2R_3$ splits into a very complicated condition of gobal degree 3 with respect to the coordinates of Ω and 28 with respect to the coordinates of P and another very simple which is that Ω must lie on the trilinear of $P \times ctP$. Note that $P \times ctP$ is called Danneels point of P in [8]. This latter condition gives a degenerate triangle with one sideline the line at infinity.

A general study seems out of reach and we shall only examine a simple (but interesting) special case namely $P = X_7$ in which case $\gamma(P)$ is the incircle of ABC.

Indeed, in this case, the two conditions above decompose and one finds that for any pseudo-pole Ω on the line passing through X_{65} and X_{513} both triangles $Q_1Q_2Q_3$ and $R_1R_2R_3$ are equilateral. This line is actually the trilinear polar of X_{2006} and the orthotransversal of X_{80}. It is also the radical axis of the circumcircle and $\mathcal{C}(X_1, X_{80})$, the circle with center the incenter X_1 passing through X_{80}. Note that these two latter points are antipodes on the Feuerbach hyperbola $\mathcal{F}$ hence $\mathcal{C}(X_1, X_{80})$ meets $\mathcal{F}$ at X_{80} and three other points which are the vertices of an equilateral triangle. Furthermore, the sidelines of this triangle are parallel to those of the Morley triangle.

There are several remarkable cubics of this type namely **K571** (see figure 8 below), **K572** , **K573** (a central cubic) , **K574** , **K575**.

Remark : when the pseudo-pole Ω is one of the common points of the line passing through X_{65}, X_{513} and the circle $\mathcal{C}(X_{65}, X_1)$, the triangles $Q_1Q_2Q_3$ and $R_1R_2R_3$ have their sidelines perpendicular to those of the Morley triangle. The equations of the two cubics are complicated.

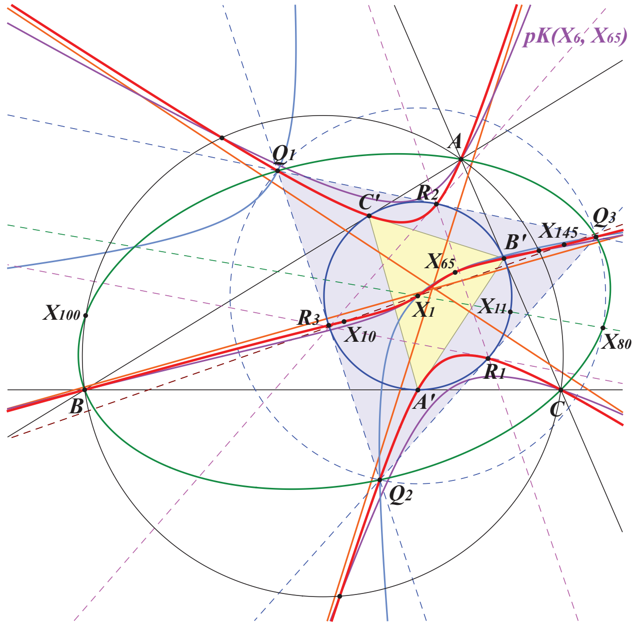

Figure 8: **K571** $= ps\mathcal{K}(X_{65}, X_7, X_1)$

3 Special cubics $ps\mathcal{K}$

We now examine under which conditions the cubic $ps\mathcal{K}(\Omega, P, \lambda)$ or $ps\mathcal{K}(\Omega, P, M)$ has certain particular characteristics. This is the object of the following paragraphs.

3.1 Equilateral cubics $ps\mathcal{K}$ or $ps\mathcal{K}_{60}$

A cubic is said to be equilateral when its has three real asymptotes forming 60° angles with one another. See [1], chapter 5.

3.1.1 Theorems and properties

Since all $ps\mathcal{K}$ with given pseudo-pole Ω and pseudo-pivot P form a pencil depending of the real number λ, there is in general no equilateral cubic $ps\mathcal{K}_{60}$ in the pencil. Indeed, a cubic is equilateral if and only if the polar conics of two distinct infinite points are rectangular hyperbolas. In the present case, these two conditions are linear with respect to λ hence the two equations have in general no common solution.

However, when we eliminate λ between these two equations, we find a condition for which there is one (and then only one) cubic $ps\mathcal{K}_{60}$ with pseudo-pole Ω and pseudo-pivot P. Naturally, the choices of Ω and P are no longer independent.

Theorem 7 *One can find one equilateral cubic $ps\mathcal{K}(\Omega, P, \lambda)$ with given pseudo-pole Ω and pseudo-pivot P if and only if*

$$\sum_{\text{cyclic}} \left(4S_A^2 - b^2c^2\right) p(v + w) = 0.$$

This condition means that

- either Ω must lie on the trilinear polar of $X_{1989} \div cP$ where cP denotes the complement of P (recall that X_{1989} is the barycentric product of the Fermat points X_{13} and X_{14}),
- or cP must lie on the trilinear polar of $X_{1989} \div \Omega$ i.e. P must lie on the anticomplement of the trilinear polar of $X_{1989} \div \Omega$.

In this case, the pseudo-isopivot P^* (remember, the common point of the tangents at A, B, C) lies on the trilinear polar of $X_{1989} \div ctP$.

It follows that when Ω (resp. P) is fixed there is a pencil of $ps\mathcal{K}_{60}$ with pseudo-pole Ω (resp. pseudo-pivot P).

Remarks :

- for a given P, there is one and only one Ω on the trilinear polar of $X_{1989} \div cP$ such that the $ps\mathcal{K}_{60}$ is a pivotal cubic. Its coordinates are complicated, see [1], §6.3.
- for a given P, there is one and only one Ω on the trilinear polar of $X_{1989} \div cP$ such that the $ps\mathcal{K}_{60}$ is a $ps\mathcal{K}_{60}^{+}$ i.e. a $ps\mathcal{K}_{60}$ with concurring asymptotes.
- these two latter cubics coincide if and only if P lies on the Neuberg cubic **K001** and then Ω lies on the cubic **K095**. The corresponding cubic is then a $p\mathcal{K}_{60}^{+}$. See [1], §6.1.

A special case

Any $ps\mathcal{K}_{60}$ with pseudo-pole X_{1989} must have its pseudo-pivot on the line at infinity. The only $p\mathcal{K}$ of this kind is **K037** $= p\mathcal{K}(X_{1989}, X_{30})$, the Tixier equilateral cubic with asymptotes concurring at the Tixier point X_{476}.

3.1.2 $ps\mathcal{K}_{60}$ with pseudo-pivot G

With $P = G$, we obtain a pencil of cubics $ps\mathcal{K}$ with coincident pseudo-pole and pseudo-isopivot on the trilinear polar of X_{1989}, a line passing through X_{51}, X_{512}, X_{1640}, $X_{30} \times X_{1989}$ and more generally through the barycentric product of X_{1989} and any infinite point. The three asymptotes form an equilateral triangle with center X_5 for any choice of the pseudo-pole.

This pencil contains several remarkable cubics :

- the pivotal cubic with pole the homothetic of X_{598} (the perspector of the Lemoine ellipse i.e. the ellipse inscribed in triangle ABC having G and K as foci) in the homothety with center G and ratio $3/2$,
- the first Musselman cubic **K026**, a central cubic passing through O and H with asymptotes concurring at X_5, these are perpendicular to the sidelines of the Morley triangle and/or parallel to those of the McCay cubic **K003**.
- the cubic with pseudo-pole X_{512} whose asymptotes form a triangle inscribed in the circle with center X_5 and radius R, hence tritangent to the nine point circle. This cubic has the same asymptotic directions as **K024** hence parallel to the sidelines of the Morley triangle.
- the cubic **K557** with pseudo-pole $X_{30} \times X_{1989}$, an axial cubic symmetric about the perpendicular bisector of OH with one asymptote parallel at X_{113} to the Euler line, the other two obtained with rotations about X_5.

All the cubics of the pencil pass through the midpoints of ABC and three other points on $\mathcal{C}(H, R)$ which are the reflections about X_5 of the common points (apart A, B, C) of the Napoleon cubic **K005** and $(\mathcal{O})$. See figure 9.

Other remarkable examples are obtained with $P = X_{69}$ giving **K071** and with $P = X_{264}$ giving **K028** both with concurring asymptotes.

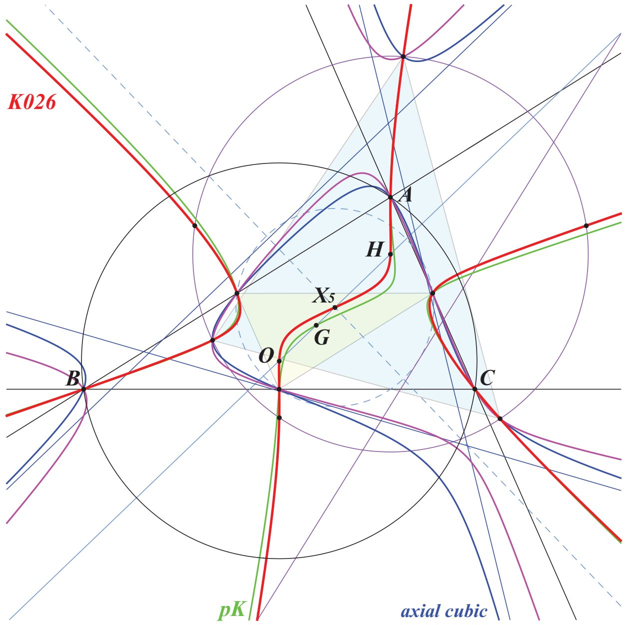

Figure 9: Equilateral cubics $ps\mathcal{K}$ with pseudo-pivot G and **K026**

3.1.3 $ps\mathcal{K}_{60}$ with pseudo-pole K

With $\Omega = K$, we obtain a pencil of cubics $ps\mathcal{K}$ with pseudo-pivot on the perpendicular at O to the Euler line and pseudo-isopivot on the rectangular circum-hyperbola passing through X_{110}. The three asymptotes form a triangle with center on the orthocentroidal circle i.e. the circle with diameter GH.

This pencil contains only one cubic with concurring asymptotes (at G) which is at the same time the only $p\mathcal{K}$ namely the McCay cubic **K003**. See figure 10.

3.2 Circular cubics $ps\mathcal{K}$

A cubic is said to be circular when it contains the circular points at infinity J_1, J_2. These two points are the only infinite points which are isogonal conjugates.

3.2.1 Theorems and properties

Since all cubics $ps\mathcal{K}(\Omega, P, \lambda)$ with given pseudo-pole Ω and given pseudo-pivot P form a pencil of cubics, there is in general no cubic of the pencil that simultaneously contains both points J_1, J_2. Indeed, when we impose that the cubic pass through these points we obtain two conditions that are generally incompatible.

However, if we eliminate λ between these two conditions, we obtain

Theorem 8 *One can find one circular cubic $ps\mathcal{K}(\Omega, P, \lambda)$ with given pseudo-pole Ω and pseudo-pivot P if and only if*

$$\sum_{\text{cyclic}} b^2c^2\, p\,(v+w) = 0 \iff \sum_{\text{cyclic}} a^2\, u(c^2q + b^2r) = 0.$$

This condition means that

- either the pseudo-pole Ω must lie on the trilinear polar of $X_6 \div cP = gcP$ in which case the pseudo-isopivot P^* lies on the trilinear polar of $gctP$,
- or P must lie on the anticomplement of the trilinear polar of $X_6 \div \Omega = g\Omega$ (or equivalently the trilinear polar of $tctg\Omega$) in which case the pseudo-isopivot lies on the circum-conic with perspector the crosspoint of X_6 and Ω .

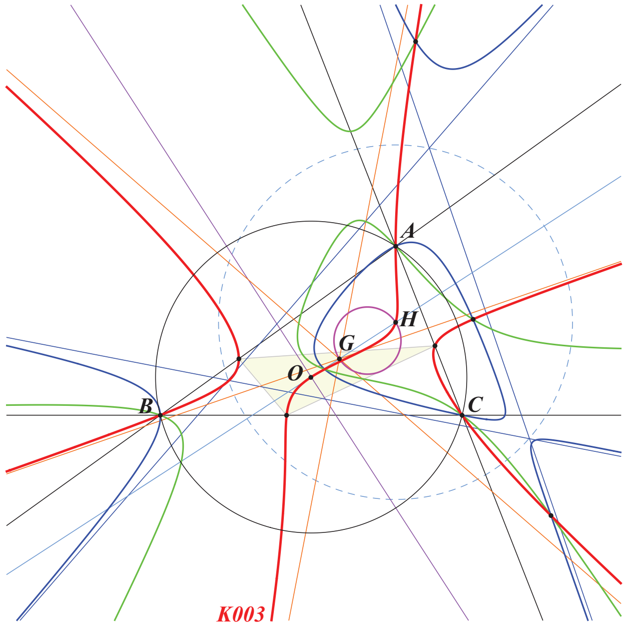

Figure 10: Equilateral cubics $ps\mathcal{K}$ with pseudo-pole K and **K003**

If we suppose that $ps\mathcal{K}$ is circular with given pseudo-pivot P, then it must contain A, B, C, P_a, P_b, P_c, J_1, J_2. This makes eight known points independent of Ω on the cubic hence it must contain a nineth point also independent of Ω. This point is M_P, the Miquel associate of P, which is the common point of the three circles AP_bP_c, BP_cP_a, CP_aP_b, each already sharing five known points with the cubic. See [8], article X_{501}.

M_P and P coincide if and only if $P = H$ thus one can find a pencil of circular cubics $ps\mathcal{K}$ with given pseudo-pole on the orthic axis and pseudo-pivot $P = H$. These are pivotal isocubics with singular focus on the nine point circle. The real asymptote envelopes the Steiner deltoid $\mathcal{H}_3$. See **CL019** in [2].

M_P and P are distinct when $P \neq H$ hence there is one and only one circular $ps\mathcal{K}$ with given $P \neq H$ and given pseudo-pole Ω on the trilinear polar of gcP.

Remarks :

- for a given $P \neq H$, there is one and only one pseudo-pole Ω on the trilinear polar of gcP such that the circular $ps\mathcal{K}$ is a pivotal cubic. This point Ω is $P \times igP$, the barycentric product of P and the inverse (in the circumcircle) of gP.
- for a given $P \neq H$, the circular cubics with Ω on the trilinear polar of gcP form a pencil and the locus of their singular focus is a circle passing through the singular focus of the corresponding $p\mathcal{K}$.
- when P lies on the line at infinity, gcP lies on the circumcircle of ABC hence its trilinear polar contains X_6. In this case, the pencil above always contains an isogonal $ps\mathcal{K}$.

On the other hand, if the pseudo-pole Ω is given and P taken on the trilinear polar of $tctg\Omega$, we obtain a pencil of cubics passing through A, B, C, J_1, J_2 and then all these cubics must have four other common points which only depend of Ω.

When $\Omega = K$, $ps\mathcal{K}$ is circular if and only if P lies on the line at infinity and, in this case, the cubic is a pivotal cubic with pivot P such as the Neuberg cubic **K001**. These form is a pencil of pivotal circular cubics hence the four points mentioned above must be the in/excenters. See **CL035** in [2] for more informations.

Now, for a given $\Omega \neq K$, there is one and only one pivotal circular cubic $p\mathcal{K}$ passing through A, B, C, J_1, J_2, J_1^*, J_2^*. See [1], §4.2 for further informations. Obviously, this $p\mathcal{K}$ contains the four cited points.

3.2.2 Circular cubics $ps\mathcal{K}$ with pseudo-pivot G

One of the most interesting examples is obtained when $P = G$ since all the cubics $ps\mathcal{K}$ pass through O (the Miquel associate of G) and have their singular focus on the circle $\mathcal{C}(O, R/2)$. The pseudo-pole Ω and the pseudo-isopivot P^* coincide and lie on the Lemoine axis, the trilinear polar of the Lemoine point K.

The only pivotal cubic of the pencil is the Droussent medial cubic **K043** $= p\mathcal{K}(X_{187}, X_2)$ with singular focus the midpoint of O, X_{111}. Its tangents at A, B, C concur at X_{187}, the common tangential of A, B, C. In the general case, the tangentials of A, B, C are distinct and each one lies on the circle passing through X_6, X_{187} and the corresponding vertex of ABC.

Any cubic of this pencil that contains M on the line at infinity also contains its isogonal conjugate gM on $(\mathcal{O})$. It meets the nine point circle $(\mathcal{N})$ on the line H, gM, at its intersection that is not the midpoint of H, gM. Here again, the real asymptote envelopes the Steiner deltoid $\mathcal{H}_3$ since it is the Simson line of the antipode of gM on $(\mathcal{O})$.

The cubic meets its real asymptote again at X which lies on the line O, gM. Note that the singular focus F of the cubic is the midpoint of O, gM. This point X lies on the symgonal quintic **Q011** since it is the intersection of the Simson line of the antipode of gM with the line O, gM.

The third point of the cubic on the line OM is also on the line HgM hence it is a point on the Lemoine cubic **K009**.

The orthic line of the cubic is the parallel to the real asymptote at X_{140}, the midpoint of X_3X_5. It meets the cubic at M and two other points whose locus is **K569** $= ps\mathcal{K}(X_6 \times X_{140}, X_2, X_3)$ passing through X_3, X_4, X_{54}, X_{140}, X_{1493}, the midpoints of ABC, the infinite points of the Napoleon cubic **K005**, the common points of $(\mathcal{O})$ and $p\mathcal{K}(X_6, X_{140})$.

Construction of the cubic

For any given point M on the line at infinity, let $\mathcal{L}$ be a variable line passing through O and let $\mathcal{L}'$ be its perpendicular at O. The circum-hyperbola passing through M and the infinite point of $\mathcal{L}'$ meets $\mathcal{L}$ at two points on the cubic.

Remarks and further properties

1. The real asymptote passes through a given point Q if and only if M is one of the three infinite points of $p\mathcal{K}(X_6, aQ)$.

2. There are three focal cubics of this type (one being always real) obtained when M is an infinite point of the Napoleon cubic **K005** hence with asymptotes passing through X_{140}, the midpoint of X_3X_5. This is the case where the real asymptote and the orthic line coincide (since they both contain X_{140}) hence F must lie on the asymptote. These cubics are central cubics with center their focus.

3. There are three axial cubics of this type (all being always real) obtained when M is an infinite point of the McCay cubic **K003** hence with asymptotes passing through X_5. This is the case where X lies on the line at infinity hence $X = M$ which turns out to be a flex on the cubic.

4. There are three cubics of this type with a flex at O obtained when M is an infinite point of the cubic $p\mathcal{K}(X_6, aX_{1092})$. aX_{1092} is the anticomplement of X_{1092} and is not mentioned in the current edition of [8]. It is the intersection of the lines X_2X_{578}, X_4X_{52}, X_5X_{1993}, $X_{20}X_{1204}$, etc.

Figure 11 shows the cubic **K570** $= ps\mathcal{K}(X_{237}, X_2, X_3)$ with focus the midpoint of X_3, X_{98} passing through X_3, X_{32}, X_{98}, X_{132}, X_{511}. Its real asymptote is the Simson line of the Steiner point X_{99} and it meets the cubic at $X = (a^4 - b^2c^2)(b^4 + c^4 - a^2b^2 - a^2c^2) : :$, not mentioned in the current edition of [8].

The table 3 shows a selection of these cubics according to their real infinite point M. See annexe at the end.

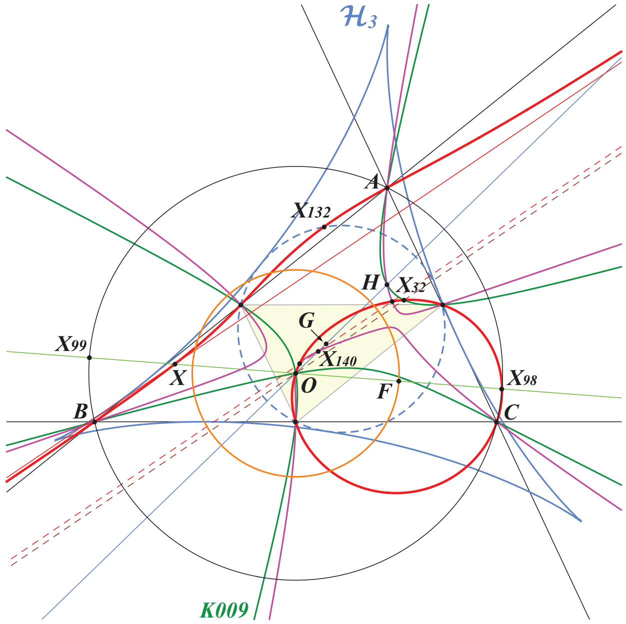

Figure 11: The circular cubic **K570** $= ps\mathcal{K}(X_{237}, X_2, X_3)$ together with **K009**

The anticomplement and the inverse in the circumcircle of any $ps\mathcal{K}$ with pseudo-pivot G are two other circular circum-cubics but they are generally not $ps\mathcal{K}$ cubics. One notable exception is obtained when the infinite point is X_{524} (that of the line GK) since the three cubics are all pivotal cubics. These are the Droussent medial cubic **K043**, the Droussent cubic **K008** and **K108** $= p\mathcal{K}(X_{32}, X_{23})$. Each contains an interesting number of centers. See list and figure 12 below.

- **K043** : X_2, X_3, X_6, X_{67}, X_{111}, X_{187}, X_{468}, X_{524}, X_{1560}, X_{2482}
- **K008** : X_2, X_4, X_{67}, X_{69}, X_{316}, X_{524}, X_{671}, X_{858}, X_{2373}
- **K108** : X_3, X_6, X_{23}, X_{25}, X_{111}, X_{187}, X_{1177}, X_{2393}, X_{2930}

3.3 Unicursal cubics $ps\mathcal{K}$

3.3.1 Theorems and properties

We already have met the Lemoine cubic **K009** $= ps\mathcal{K}(X_{184}, X_2, X_3)$ and the third Musselman cubic **K028** $= ps\mathcal{K}(X_4, X_{264}, X_3)$, both nodal equilateral cubics with respective nodes O, H and with perpendicular nodal tangents, each being the isogonal transform of the other.

We now intend to characterize all the cubics $ps\mathcal{K}(\Omega, P, M)$ with given pseudo-pivot P or given pseudo-pole Ω, passing through a given point M which has to be a node on the cubic. We suppose that M is not on the sidelines of ABC nor on a cevian of P to avoid a degenerate cubic.

We consider a variable line $\mathcal{L}_t$ passing through M and the infinite point $1 + t : t : -1 - 2t$. We express that $\mathcal{L}_t$ meets the cubic at M twice and obtain a polynomial of degree 1 in t whose coefficients are linear with respect to the coordinates of P and Ω. This gives two conditions we can solve either in P or in Ω.

We obtain

Theorem 9 *For given points M, P or M, Ω, there is a unique $ps\mathcal{K}(\Omega, P, M)$ with node at M. Furthermore,*
– when P is given then the pseudo-pole Ω must be the pole of the isoconjugation that swaps M and P/M,
– when Ω is given then the pseudo-pivot P must be the isoconjugate of the crosspoint of M and M^*.

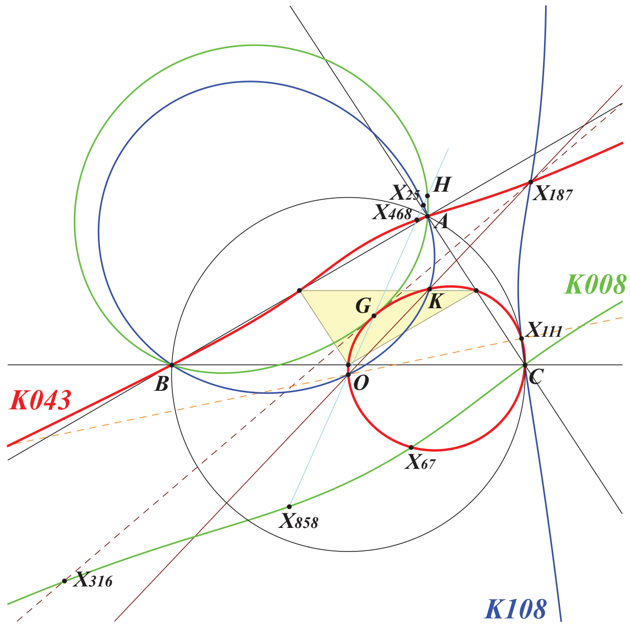

Figure 12: The circular cubics **K043**, **K008** and **K108**

With $P = u : v : w$ and $M = l : m : n$ suitably chosen as mentioned above, the nodal cubic has equation :

$$\begin{aligned}&\sum_{\text{cyclic}} u\,x(-l\,vw + m\,wu + n\,uv)(n^2y^2 - m^2z^2)\\ &+2\left(\sum_{\text{cyclic}}(nv - mw)x\right)\left(\sum_{\text{cyclic}} l^2\,vw\,yz\right) = 0,\end{aligned} \tag{6}$$

and the equation of the nodal tangents is :

$$\sum_{\text{cyclic}} mn(nv - mw)\left((-lvw + mwu + nuv)\,x^2 + 2\,l\,u^2\,yz\right) = 0, \tag{7}$$

this being evidently the equation of the polar conic of M.

Note that if the cubic $ps\mathcal{K}$ is a nodal cubic with node M then its isoconjugate is another nodal cubic with node M^*.

3.3.2 Construction of a singular $ps\mathcal{K}$

When a cubic $ps\mathcal{K}$ is singular, its construction considerably simplifies and can be realized with a ruler only. See [3], §5.

Figure 13 shows two nodal pseudo-isogonal nodal cubics with pseudo-pivot X_{83} and nodes G and K. Each is the isogonal transform of the other and both cubics are nodal parents of the corresponding pivotal cubic $p\mathcal{K}(X_6, X_{83})$.

In general, the nodal tangents are not perpendicular unless

Theorem 10 *The nodal cubic $ps\mathcal{K}$ with node M has perpendicular nodal tangents if and only if its pseudo-pivot P lies on $p\mathcal{K}_P$, the $p\mathcal{K}$ with isopivot M and pivot P_M, the perspector of the inconic $\mathcal{I}_F(M)$ with focus M. In this case, the pseudo-pole of $ps\mathcal{K}$ lies on $p\mathcal{K}_\Omega$, the $p\mathcal{K}$ with pivot $M \times M^\perp$ and pole M^4.*

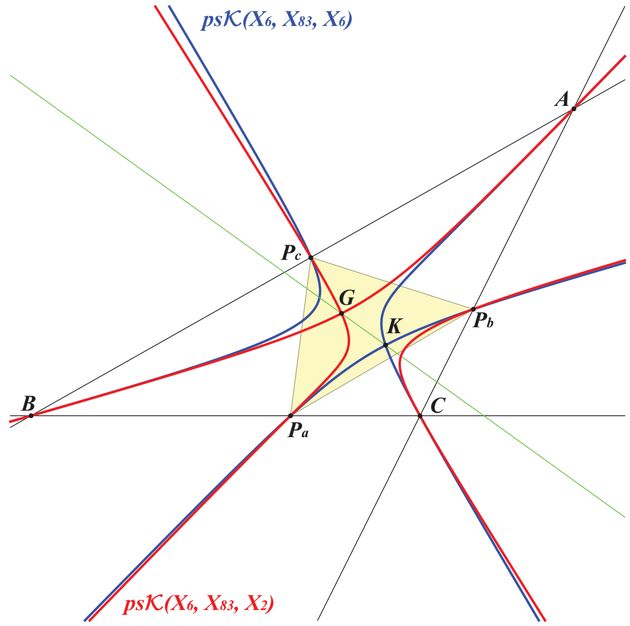

Figure 13: The nodal cubics $ps\mathcal{K}(X_6, X_{83}, X_2)$ and $ps\mathcal{K}(X_6, X_{83}, X_6)$

Here M^4 is the fourth barycentric power of M and the point $M^\perp$ is the orthocorrespondent of M. See definition and properties in [4].

Note that $p\mathcal{K}_P$ contains, apart its pivot P_M and its isopivot M, the point $M^\perp$ and its isoconjugate.

3.3.3 Two special cases

- With $P = G$ we obtain the class **CL033** of Deléham cubics in [2]. The pseudo-pole is now that of the isoconjugation that swaps M and the center G/M of the circum-conic with perspector M. All these cubics contain the vertices of the medial triangle. The nodal tangents are perpendicular when M lies on the Thomson cubic **K002**.

  With $M = X_1$, X_3, X_6, X_9 we find the cubics **K259**, **K009**, **K260**, **K220** respectively.

- When M lies on the Darboux cubic **K004**, the pedal triangle of M is a cevian triangle with perspector P on the Lucas cubic **K007**. In this case, the nodal cubic $ps\mathcal{K}$ with node M and pseudo-pivot P is a Lemoine generalized cubic. See [3].

  With $P = X_1$, X_3, X_{20} we find the cubics **K360**, **K009**, **K041** respectively.

3.3.4 Other examples

Example 1 : With $M = X_1$ we have $P_M = X_7$ hence $p\mathcal{K}_P$ is **K365** $= p\mathcal{K}(X_{57}, X_7)$ and $p\mathcal{K}_\Omega$ is $p\mathcal{K}(X_{32}, X_{56})$. This gives several interesting cubics such as

- **K360** $= ps\mathcal{K}(X_{56}, X_7, X_1)$ (a Lemoine generalized cubic, see [3]), passing through X_1, X_4, X_{56}, X_{145}, X_{218}, X_{279}, X_{1433}, the vertices of the intouch triangle,
- **K259** $= ps\mathcal{K}(X_{55}, X_2, X_1)$ passing through X_1, X_3, X_8, X_{220}, X_{277}, X_{3160}, the vertices of the medial triangle,
- $ps\mathcal{K}(X_{3052}, X_{57}, X_1)$ passing through X_1, X_{1420}, X_{2136}.

Example 2 : With $M = X_4$ we have $P_M = X_{264}$ hence $p\mathcal{K}_P$ is $p\mathcal{K}(X_{2052}, X_{264})$ passing through X_2, X_4, X_{92}, X_{253}, X_{264}, X_{273}, X_{318}, X_{342}, X_{2052}. $p\mathcal{K}_\Omega$ is complicated. The most remarkable corresponding cubics are

- **K028** $= ps\mathcal{K}(X_4, X_{264}, X_4)$, the third Musselman cubic,
- $ps\mathcal{K}(X_{1118}, X_{273}, X_4)$ passing through X_4, X_{34}, X_{84}, X_{158}, X_{1847},
- $ps\mathcal{K}(X_4^3, X_{2052}, X_4)$ passing through X_4, X_{24}, X_{64}, X_{1093}, X_{1118}, X_{2207}. Here X_4^3 is the barycentric cube of the orthocenter.

The following table gives a selection of nodal cubics $ps\mathcal{K}$.

cubic	$ps\mathcal{K}$	node	centers on the cubic	remark
K009	$ps\mathcal{K}(X_{184}, X_2, X_3)$	X_3	X_3, X_4, X_{32}, X_{56}, X_{1147}	
K028	$ps\mathcal{K}(X_4, X_{264}, X_3)$	X_4	X_3, X_4, X_8, X_{76}, X_{847}	$ps\mathcal{K}^+_{60}$
K041	$ps\mathcal{K}(X_{20}, X_{69}, X_4)$	X_{20}	X_4, X_{20}, X_{279}, X_{280}	
K220	$ps\mathcal{K}(X_{55}, X_2, X_6)$	X_9	X_6, X_7, X_9, X_{173}, X_{268}, X_{281}, X_{3161}	
K257	$ps\mathcal{K}(X_{69}, X_{76}, X_6)$	X_{69}	X_6, X_7, X_{69}, X_{264}	$ps\mathcal{K}^+$
K259	$ps\mathcal{K}(X_{55}, X_2, X_1)$	X_1	X_1, X_3, X_8, X_{220}, X_{277}, X_{3160}	
K260	$ps\mathcal{K}(X_{184}, X_2, X_6)$	X_6	X_6, X_{69}, X_{206}, X_{219}, X_{478}, X_{577}, X_{1249}, X_{2165}	
K360	$ps\mathcal{K}(X_{56}, X_7, X_1)$	X_1	X_1, X_4, X_{56}, X_{145}, X_{218}, X_{279}, X_{1433}	
K429	$ps\mathcal{K}(X_{1974}, X_4, X_6)$	X_6	X_6, X_{66}, X_{193}, X_{393}, X_{571}, X_{608}, X_{1974}, X_{2911}	
K555	$ps\mathcal{K}(X_4, X_{264}, X_2)$	X_2	X_2, X_{25}, X_{278}, X_{1073}, X_{1993}, X_{2052}, X_{3190}	
	$ps\mathcal{K}(X_1, X_{86}, X_1)$	X_1	X_1, X_{10}, X_{21}, X_{274}	$ps\mathcal{K}^+$
	$ps\mathcal{K}(X_3, X_{95}, X_3)$	X_3	X_3, X_5, X_{54}, X_{276}	$ps\mathcal{K}^+$
	$ps\mathcal{K}(X_6, X_{83}, X_6)$	X_6	X_6, X_{141}, X_{308}, X_{1176}	$ps\mathcal{K}^+$
	$ps\mathcal{K}(X_7, X_{85}, X_7)$	X_7	X_7, X_9, X_{75}, X_{77}, X_{342}	$ps\mathcal{K}^+$
	$ps\mathcal{K}(X_8, X_{75}, X_8)$	X_8	X_1, X_8, X_{85}, X_{271}, X_{318}	$ps\mathcal{K}^+$

Remark : $ps\mathcal{K}(M, tcM, M)$ is always a $ps\mathcal{K}^+$ i.e. a nodal cubic with asymptotes concurring at the midpoint of GM. The nodal tangents are perpendicular if and only if M lies on the Lucas cubic **K007**.

3.4 Cuspidal cubics $ps\mathcal{K}$

Theorem 11 *The singular cubic $ps\mathcal{K}(M \times P/M, P, M)$ has a cusp at M if and only if M lies on a cubic with node P which is the transform of $c\mathcal{K}(\#P, P)$ under the homology with pole P, with axis the trilinear polar of P, which maps the cevian triangle $P_aP_bP_c$ of P to ABC.*

Furthermore, the cuspidal tangent is tangent to the inconic with perspector P.

Recall that $c\mathcal{K}(\#P, P)$ is the conico-pivotal cubic with singularity at P and root P. See [1] for more informations.

An easy construction of $c\mathcal{K}(\#P, P)$ derives from the fact that it is the locus of the intersection of a tangent at X on the circum-conic with perspector P with the trilinear polar of X.

With $P = u : v : w$ the homology above is given by $x : y : z \mapsto u\left(-\dfrac{x}{u} + \dfrac{y}{v} + \dfrac{z}{w}\right) : : .$

Figure 14 shows **K228** $= c\mathcal{K}(\#X_1, X_1)$ and its transform. Here, the axis of homology is the trilinear polar of the incenter X_1 namely the antiorthic axis $\mathcal{A}$. When m is taken on **K228**, the line Bm meets $\mathcal{A}$ at n. The lines mX_1 and nP_b intersect at M on the cubic.

Since X_{2087} lies on the cubic, one can find a cuspidal cubic $ps\mathcal{K}$ with pseudo-pivot X_1 and cusp at X_{2087}. This is **K560**. See figure 15.

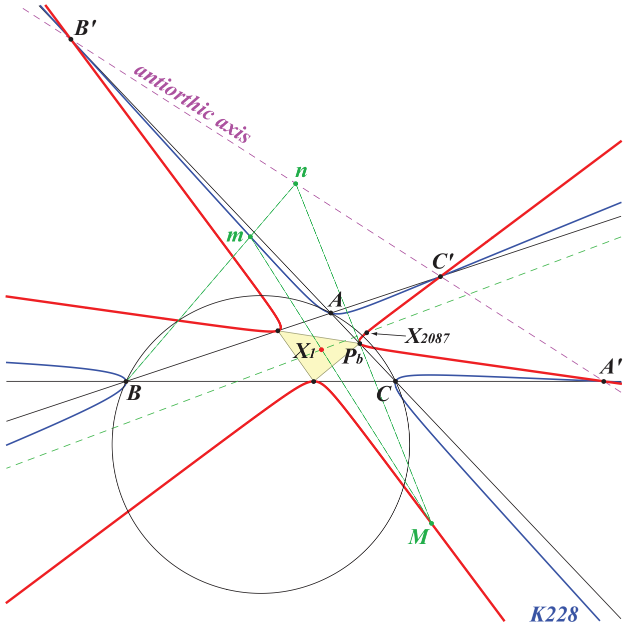

Figure 14: Locus of M such that $ps\mathcal{K}$ is a cuspidal cubic

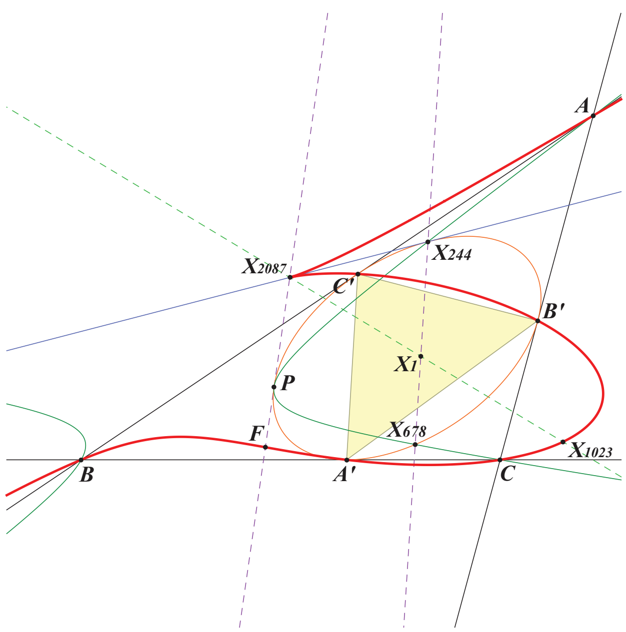

Figure 15: **K560**, a cuspidal cubic $ps\mathcal{K}$ with pseudo-pivot X_1

The most prolific example is obtained when $P = X_2$ since M is a point on **K219**, the complement of the Tucker nodal cubic **K015** $= c\mathcal{K}(\#X_2, X_2)$. **K219** contains X_2, X_{1645}, X_{1646}, X_{1647}, X_{1648}, X_{1649}, X_{1650} and more generally the tripolar centroid of any point Q on the line at infinity. The cuspidal tangent is the tangent at Q^2 (barycentric square) to the Steiner inscribed ellipse. The hessian of the cuspidal cubic decomposes into the cuspidal tangent (counted twice) and another line which is the second tangent drawn from M to the Steiner inscribed ellipse. This latter line must contain the real inflexion point which turns out to be the barycentric square of the tripolar centroid of the infinite point of the conjugated diameter of the line GM in the Steiner ellipse.

3.5 Central cubics $ps\mathcal{K}$ or $ps\mathcal{K}^{++}$

3.5.1 Theorems and properties

We fix the point M and seek cubics $ps\mathcal{K}(\Omega, P, M)$ that are central cubics with center M. Let M_a, M_b, M_c be the reflections of A, B, C about M. Recall that P_a, P_b, P_c are the vertices of the cevian triangle of P.

A central $ps\mathcal{K}(\Omega, P, M)$ with center M must contain M_a, M_b, M_c. This gives three conditions linear with respect to the coordinates of Ω and P which are generally incompatible. When we eliminate the coordinates of Ω or P, we obtain

Proposition 4 *The cubic $ps\mathcal{K}(\Omega, P, M)$ is a central cubic with center M if and only if*

1. *its pseudo-pivot P lies on the pivotal cubic $p\mathcal{K}(M \times taM, taM)$ here denoted by $p\mathcal{K}(P)$,*
2. *its pseudo-pole Ω lies on the pivotal cubic $p\mathcal{K}(M^2 \times G/M, M)$ here denoted by $p\mathcal{K}(\Omega)$.*

Now, in order to find the correspondence between the pseudo-pivot P and the pseudo-isopivot P^*, we first examine the circum-cubic passing through M_a, M_b, M_c, P_a, P_b, P_c. This cubic has always concurring tangents at A, B, C which meet at $(G/M) \times a(P \times aM)$ thus, with P on $p\mathcal{K}(P)$, we have

Proposition 5 *When $ps\mathcal{K}(\Omega, P, M)$ is a central cubic with center M, its pseudo-isopivot P^* is $(G/M) \times a(P \times aM)$ and lies on the pivotal cubic $p\mathcal{K}((G/M)^2, M)$ here denoted by $p\mathcal{K}(P^*)$.*

Remarks :

1. The two cubics $p\mathcal{K}(P)$ and $p\mathcal{K}(\Omega)$ contain M with the same tangent passing through $M \times taM$. The cubic $p\mathcal{K}(P^*)$ also contains M with a tangent passing through G.
2. The isoconjugation with fixed point M swaps globally these two cubics $p\mathcal{K}(P)$ and $p\mathcal{K}(\Omega)$ but does not transform the pseudo-pivot P of the sought central cubic $ps\mathcal{K}(\Omega, P, M)$ into the corresponding pseudo-pole.
3. Naturally, the knowledge of P on $p\mathcal{K}(P)$ determines only one possible cubic since we already know seven points on the cubic $ps\mathcal{K}(\Omega, P, M)$. Indeed, in this case, this gives six more points on the cubic namely P_a, P_b, P_c and their reflections about M.

In conclusion,

- if P does not lie on $p\mathcal{K}(P)$, there is no central $ps\mathcal{K}$ with center M and pseudo-pivot P,
- if P lies on $p\mathcal{K}(P)$, there is a unique central $ps\mathcal{K}$ with center M and it is a (proper) pivotal cubic if and only if $P = aaM$ giving the cubic $p\mathcal{K}(G/M, aaM)$.

P on $p\mathcal{K}(P)$	Ω on $p\mathcal{K}(\Omega)$	P^* on $p\mathcal{K}(P^*)$	remarks about $ps\mathcal{K}(\Omega, P, M)$
M	M^2	M	union of lines AM, BM, CM
taM	M	G/M	
$M \times taM$	$M \times G/M$	$G/M \times aM$	
aaM	G/M	$G/M \div aaM$	pivotal cubic
$M \times taM \div aaM$	$G/M \times \widetilde{G}$	$\widetilde{G} \times aaM \times (aM)^2$	
G	$M/(G/M)$	$M/(G/M)$	
$\widetilde{G}$	$M^2 \div aaM$	$M^2 \div (\widetilde{G} \times aaM)$	

The following table gives some correspondences between P on $p\mathcal{K}(P)$, Ω on $p\mathcal{K}(\Omega)$ and P^* on $p\mathcal{K}(P^*)$. Note : $\widetilde{G}$ is the tangential of G in $p\mathcal{K}(P)$ and M^2 is the barycentric square of M.

Remarks :

1. When $P = aaM$ and $\Omega = G/M$, the cubic is a pivotal cubic.
2. Since G lies on $p\mathcal{K}(P)$, one can always get a central $ps\mathcal{K}$ for any center M and this cubic obviously contains the vertices of the medial triangle.
3. The cubics obtained with $P = G$ and $P = \widetilde{G}$ also have the same tangent at M which passes through $M/(G/M)$.
4. The cubics obtained with $P = taM$ and $P = M \times taM$ have the same tangent at M, that of $p\mathcal{K}(P)$ and $p\mathcal{K}(\Omega)$.
5. $p\mathcal{K}(P)$ is said to be a perspector cubic and it is closely related to the central pivotal cubic $p\mathcal{K}(G/M, aaM)$ we have met above. If A', B', C' are the reflections of A, B, C about M then for any point P let A_P, B_P, C_P be the traces of the lines PA', PB', PC' on the sidelines of ABC. The triangles ABC and $A_PB_PC_P$ are perspective if and only if P lies on $p\mathcal{K}(G/M, aaM)$ and the locus of the perspector is $p\mathcal{K}(P)$.

3.5.2 Central cubics $ps\mathcal{K}$ with center O

We take $M = O$ thus all the central cubics $ps\mathcal{K}$ contain the antipodes A', B', C' of A, B, C on $(\mathcal{O})$.

The corresponding cubics $p\mathcal{K}(P)$, $p\mathcal{K}(\Omega)$ and $p\mathcal{K}(P^*)$ contain many triangle centers. Indeed,

- $p\mathcal{K}(P)$ is the Darboux perspector cubic **K099** $= p\mathcal{K}(X_{394}, X_{69})$ passing through X_2, X_3, X_{20}, X_{63}, X_{69}, X_{77}, X_{78}, X_{271}, X_{394},
- $p\mathcal{K}(\Omega)$ is **K576** $= p\mathcal{K}(X_3 \times X_{184}, X_3)$ passing through X_3, X_6, X_{48}, X_{154}, X_{184}, X_{212}, X_{577}, X_{603}, X_{2188}.
- $p\mathcal{K}(P^*)$ is **K172** $= p\mathcal{K}(X_{32}, X_3)$ passing through X_3, X_6, X_{25}, X_{55}, X_{56}, X_{64}, X_{154}, X_{198}, X_{1033}, X_{1035}, X_{1436}.

This gives a good selection of central cubics with center O and in particular the Darboux cubic **K004**, the only $p\mathcal{K}$ of the family.

The table below gives the correspondence between the points P and Ω with the corresponding centers on the cubics.

Recall that the Darboux cubic **K004** contains X_1, X_3, X_4, X_{20}, X_{40}, X_{64}, X_{84}, X_{1490}, X_{1498}, X_{2130}, X_{2131}, X_{3182}, X_{3183}, X_{3345}, X_{3346}, X_{3347}, X_{3348}, X_{3353}, X_{3354}, X_{3355}, X_{3472}, X_{3473}.

Figure 16 shows the three catalogued cubics of the list. They contain O, H, $L = X_{20}$ hence they belong to a same pencil of cubics. The equilateral central cubic **K080** is a member of this pencil but it is not a $ps\mathcal{K}$.

P on **K099**	Ω on **K576**	$ps\mathcal{K}$ / centers on $ps\mathcal{K}$
X_2	X_{154}	**K426** / X_3, X_4, X_{20}
X_3	X_{577}	union of the cevian lines of O
X_{20}	X_6	**K004**, the Darboux cubic
X_{63}	X_{212}	X_1, X_3, X_{40}
X_{69}	X_3	**K443** / X_3, X_4, X_{20}
X_{77}	$X_{77} \times X_{198}$	X_1, X_3, X_{40}
X_{78}	X_{48}	X_3, X_{84}, X_{1490}
X_{271}	?	X_3, X_{84}, X_{1394} , X_{1490}
X_{394}	X_{184}	X_3, X_{64}, X_{1498}
$X_{77} \times X_{329}$	X_{603}	X_3, X_{3182}
?	X_{2188}	X_3

Table 1: Central cubics $ps\mathcal{K}$ with center O

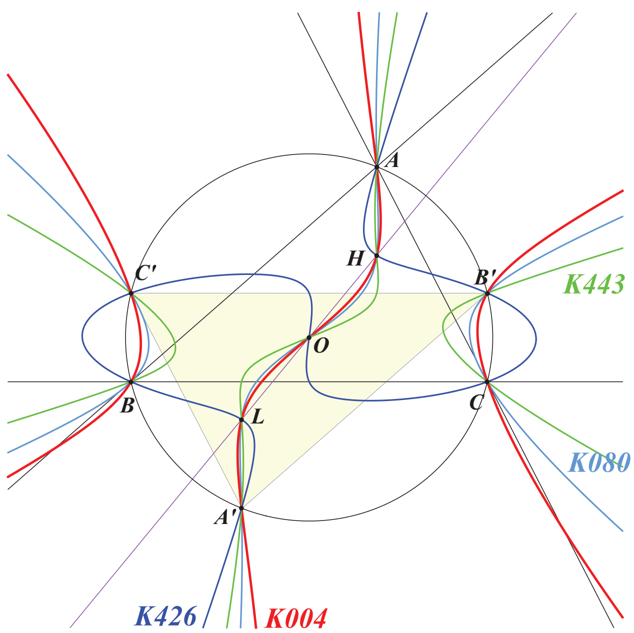

Figure 16: The central cubics **K004**, **K426**, **K443**

3.6 Axial cubics $ps\mathcal{K}$

We already have met several axial cubics $ps\mathcal{K}$ i.e. cubics with an axis of symmetry namely **K556**, **K557** (an equilateral cubic), **K578**, and also a triad of circular cubics with pseudo-pivot G.

Now, given a line $\mathcal{L}$ supposed to be an axis of symmetry of a sought cubic $ps\mathcal{K}$, we intend to characterize its other elements. One obvious but trivial cubic is the union of the perpendiculars at A, B, C to $\mathcal{L}$.

Let F' be the point at infinity of $\mathcal{L}$ and let T be its trilinear pole.

The first and easy thing to observe is that the cubic must contain the infinite point F of any perpendicular to $\mathcal{L}$ and furthermore the polar conic of F must decompose into two perpendicular lines namely the axis of symmetry $\mathcal{L}$ and the (always real) asymptote that contains F.

A very tedious computation gives the following

Theorem 12 *There are infinitely many axial cubics $ps\mathcal{K}$ with given axis of symmetry $\mathcal{L}$. Their pseudo-poles lie on $p\mathcal{K}(F^3 \times T, F \times T)$ and their pseudo-pivots lie on $p\mathcal{K}(F \times S, S)$ where S is a point described below.*

If $\mathcal{L}$ has line-coordinates $U : V : W$ then the isotomic conjugate of S is tS with first barycentric coordinate $U[2S_AVW - (-a^2U^2 + b^2V^2 + c^2W^2)]$. This point tS lies on the bicevian hyperbola $\mathcal{H}(tF, tF')$ that contains F, F', the Nagel point X_8 and its three extraversions.

Construction of S

The isogonal transform of $\mathcal{L}$ is the circum-conic with perspector gT and center G/gT. The line passing through this center and the pole of $\mathcal{L}$ in the circum-conic meets $\mathcal{L}$ at M and then S is taM.

Note that this point S is the pseudo-pivot of the axial $ps\mathcal{K}$ with pseudo-pole $F \times T$. In other words, $ps\mathcal{K}(T \times F, S, F)$ is always an axial cubic.

Example 1 : $\mathcal{L}$ is the Euler line

Here we have $F = X_{523}$, $F' = X_{30}$, $T = X_{648}$, $S = X_{264}$.

The pseudo-pole Ω of the axial cubics $ps\mathcal{K}$ lies on $p\mathcal{K}(X_{115} \times X_4, X_4)$ passing through X_4, X_{115}, X_{512}, X_{2039}, X_{2040} and the pseudo-pivot P lies on $p\mathcal{K}(X_{523} \times X_{264}, X_{264})$ passing through X_{264}, X_{338}, X_{523}, X_{648}.

With $\Omega = X_4$ we must have $P = X_{264}$ and the corresponding cubic is **K556** $= ps\mathcal{K}(X_4, X_{264}, X_{523})$.

Another interesting cubic is $ps\mathcal{K}(X_{512}, X_{338}, X_{523})$ and these two cubics have the same real asymptote namely the line X_{338}, X_{523}. See figure 17.

Example 2 : $\mathcal{L}$ is the perpendicular bisector of OH i.e. the line X_5, X_{523}

Here we have $F = X_{30}$, $F' = X_{523}$, $T = X_{94}$, $S = X_{264}$.

The pseudo-pole Ω of the axial cubics $ps\mathcal{K}$ lies on $p\mathcal{K}(X_{94} \times X_{30}^3, X_{94} \times X_{30})$ passing through X_{3163} and the pseudo-pivot P lies on $p\mathcal{K}(X_{30} \times X_{264}, X_{264})$ passing through X_2, X_4, X_{30}, X_{94}, X_{264}, X_{1300}, X_{1494}.

With $\Omega = X_{30} \times X_{1989}$ we must have $P = X_2$ and the corresponding cubic is the equilateral cubic **K557** $= ps\mathcal{K}(X_{30} \times X_{1989}, X_2, X_{30})$.

4 Pencils and nets of cubics $ps\mathcal{K}$

We shall now combine the results of the previous section and examine some interesting choices of Ω and P (or P and M) leading to pencils or nets containing simultaneously several special cubics.

4.1 Pencils of cubics $ps\mathcal{K}$ with given Ω and P (or given P and P^*)

First, let us recall that all the cubics with given Ω and P form a pencil that may be generated by $p\mathcal{K}(\Omega, P)$ and the union of the sidelines of ABC. In general, this pencil does not contain a circular cubic nor an equilateral cubic since each is characterized by two supplementary conditions giving a system in general without any solution.

On the other hand, we have seen that,

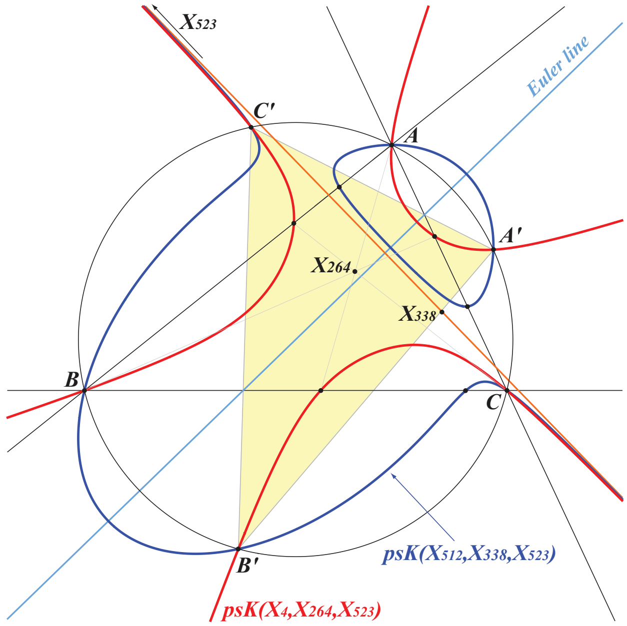

Figure 17: Two axial cubics with axis the Euler line

- when Ω lies on the trilinear polar of $gcP = X_6 \div cP$ then there is a pencil of circular cubics $ps\mathcal{K}$ with pseudo-pivot P and passing through the Miquel associate of P which is the nineth base point of the pencil,
- when Ω lies on the trilinear polar of $X_{1989} \div cP$ then there is a pencil of equilateral cubics $ps\mathcal{K}$ with pseudo-pivot P.

Since these two trilinear polars are distinct and meet at $\Omega_o = X_{512} \div cP$, we obtain :

Theorem 13 *For any pseudo-pivot P, there is a pencil of cubics $ps\mathcal{K}$ with pseudo-pole Ω_o that contains one circular cubic and one equilateral cubic.*

Remarks

1. The pencil obviously also contains $p\mathcal{K}(\Omega_o, P)$ which coincide with the circular cubic when P lies on a circular cubic $\mathcal{K}_c$ and with the equilateral cubic when P lies on an equilateral cubic $\mathcal{K}_e$.

   These two cubics are both circum-cubics passing through the vertices of the antimedial triangle $G_aG_bG_c$ with the same tangents perpendicular to the Brocard axis. $\mathcal{K}_c$ also contains X_4, X_{523}, X_{925} and its singular focus is X_{265}. The asymptotes of $\mathcal{K}_e$ are the sidelines of the tangential triangle. See figure 18.

   It is clear that the pencil cannot contain a cubic that is at the same time circular and equilateral hence any $p\mathcal{K}(\Omega_o, P)$ with P on the two cubics $\mathcal{K}_c$ and $\mathcal{K}_e$ must decompose.

2. The circular cubic always contains X_{523} hence its real asymptote is perpendicular to the Euler line.

3. The equilateral cubic has always the same asymptotic directions, that of **K024**, hence the asymptotes are parallel to the sidelines of the Morley triangle.

   The (equilateral) triangle formed by these asymptotes is centered on the Euler line when P lies on **K242** $= p\mathcal{K}(X_2, X_{99})$ passing through X_2, X_{99}, X_{523}, X_{1113}, X_{1114}.

We illustrate the theorem above with two examples of pencils, each being the isogonal transform of the other.

Example 1 : $P = G$ hence $\Omega_o = X_{512}$, see figure 19.

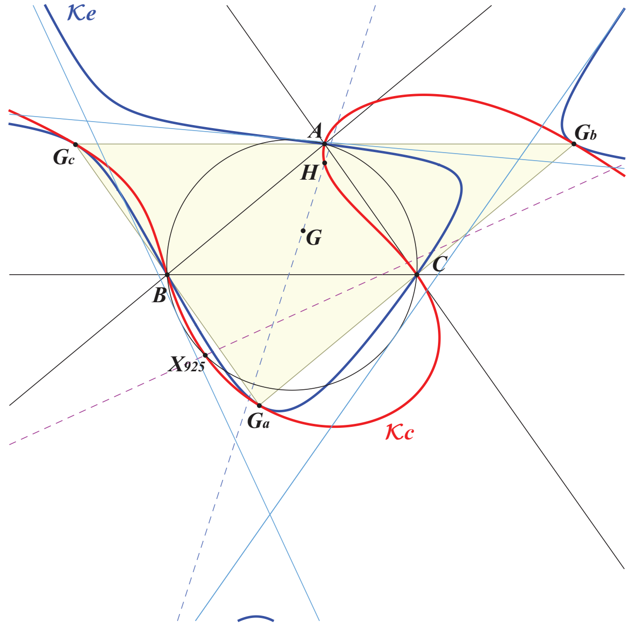

Figure 18: The cubics $\mathcal{K}_c$ and $\mathcal{K}_e$

- the pivotal cubic is $p\mathcal{K}(X_{512}, X_2)$ passing through X_2, X_{512}, X_{670}, X_{1084},
- the circular cubic is **K567** $= ps\mathcal{K}(X_{512}, X_2, X_3)$ passing through X_3, X_{110}, X_{136}, X_{523} with singular focus X_{1511},
- the equilateral cubic does not contain any known center but its asymptotes form an equilateral triangle with center X_5.

Example 2 : $P = X_{99}$ hence $\Omega_o = X_{110}$, see figure 20.

- the pivotal cubic is $p\mathcal{K}(X_{110}, X_{99})$ passing through X_6, X_{99}, X_{699},
- the circular cubic is **K568** $= ps\mathcal{K}(X_{110}, X_{99}, X_4)$ passing through X_4, X_{110}, X_{523} with singular focus X_{399},
- the equilateral cubic does not contain any known center but its asymptotes form an equilateral triangle with center X_3.

4.2 Nets of cubics $ps\mathcal{K}$ with given P and M

We know consider the family of cubics of the form $ps\mathcal{K}(\Omega, P, M)$ with given P and $M \neq P$. These cubics form a net of circum-cubics passing through the cevian points P_a, P_b, P_c of P and through M.

This net may be generated by three decomposed cubics each consisting in two sidelines of ABC and the line passing through M and the trace of P on the third sideline of ABC.

In general, this net contains :

- one nodal cubic with node M when $\Omega = M \times P/M$,
- one equilateral $ps\mathcal{K}$,
- one circular $ps\mathcal{K}$ unless M is the Miquel associate of P in which case there is a pencil of circular cubics with pole on the trilinear polar of gcP,

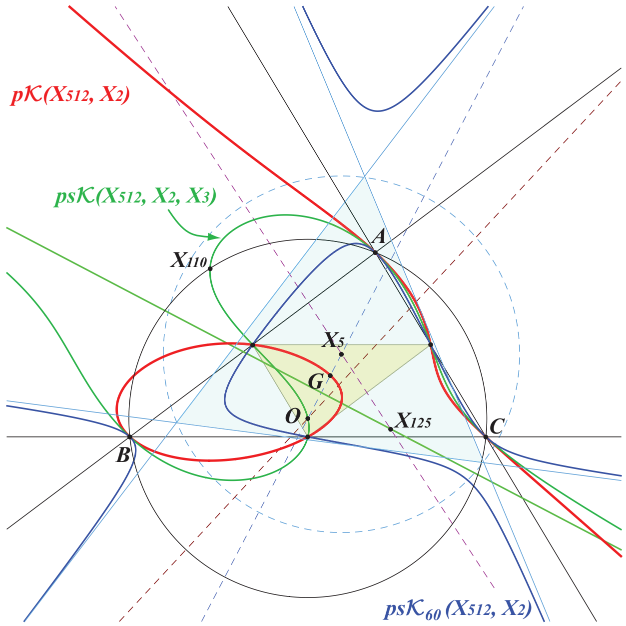

Figure 19: The pencil of cubics $p\mathcal{K}(X_{512}, X_2, \lambda)$

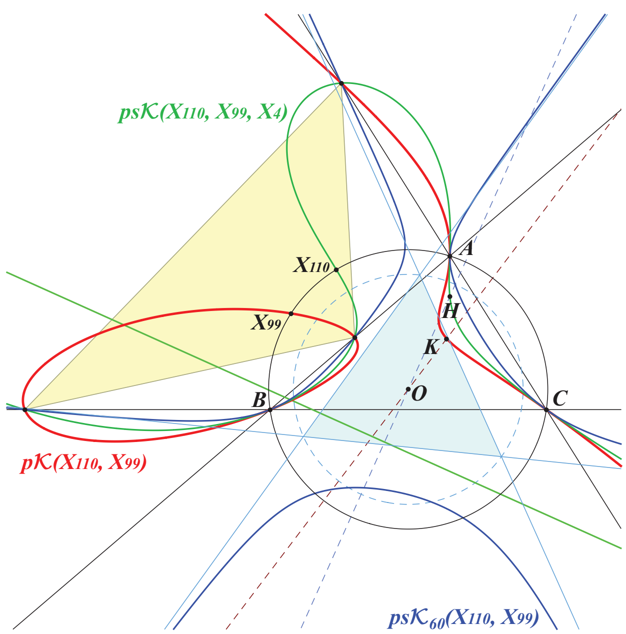

Figure 20: The pencil of cubics $p\mathcal{K}(X_{99}, X_{110}, \lambda)$

- a pencil of pivotal cubics with pivot P when Ω lies on the line passing through M^2 (giving a decomposed cubic) and $P \times M$ (giving $p\mathcal{K}(P \times M, P)$), these two cubics generating the pencil and then every cubic also contains P/M,

- a family of central cubics with center M if and only if P lies on $p\mathcal{K}(P) = p\mathcal{K}(M \times taM, taM)$ in which case Ω lies on $p\mathcal{K}(\Omega) = p\mathcal{K}(M^2 \times G/M, M)$.

- other cubics with concurring asymptotes when Ω lies on a cubic with rather complicated equation.

Naturally, some of these cubics may coincide. For example, the equilateral (or the circular cubic) can be pivotal or central, etc.

We present several other examples with interesting choices of P and M.

4.2.1 Example 3 : $P = H$ and $M = X_5$, see figure 21

These are the "orthic" cubics $ps\mathcal{K}$ passing through the vertices H_a, H_b, H_c of the orthic triangle.

The pivotal cubics have their pole on the Brocard axis of the orthic triangle passing through X_5, X_{53}, X_{216}, X_{233}, X_{3199}. Each pole is the barycentric product of H and a point on the Euler line of the orthic triangle.

Ω	cubic or centers	comment
X_5	X_2, X_4, X_5, X_{52}, X_{193}, X_{343}	$p\mathcal{K}$
X_{51}	X_2, X_5, X_{25}, X_{53}, X_{155}, X_{2165}	
X_{53}	**K049**, McCay orthic	$p\mathcal{K}$, equilateral
X_{216}	**K044**, Darboux orthic	$p\mathcal{K}$, central
X_{233}	X_4, X_5, X_{52}, X_{140}	$p\mathcal{K}$
X_{3199}	**K350**, Thomson orthic	$p\mathcal{K}$
$X_5 \times X_{186}$	**K050**, Neuberg orthic	$p\mathcal{K}$, circular
$X_4 \times X_{52}$	**K415**, Orthocubic orthic	$p\mathcal{K}$
$X_4 \times X_{143}$	**K416**, Napoleon orthic	$p\mathcal{K}$
$X_5 \times X_{52}$	X_5, X_{24}, X_{155}	nodal
$X_4 \times X_{973}$	X_4, X_5, X_{52}, X_{973}, X_{1166}	$p\mathcal{K}$
$X_4 \times X_{1209}$	X_4, X_5, X_{52}, X_{1209}, X_{1594}	$p\mathcal{K}$
$X_4 \times X_{1568}$	X_4, X_5, X_{30}, X_{52}, X_{113}, X_{1568}	$p\mathcal{K}$

Table 2: $ps\mathcal{K}(\Omega, X_4, X_5)$

4.2.2 Example 4 : $P = X_{264}$ and $M = O = X_3$, see figure 22

In this example, we meet

- the third Musselman cubic **K028** $= ps\mathcal{K}(X_4, X_{264}, X_3)$, an equilateral nodal cubic with perpendicular nodal tangents and concurring asymptotes,

- the central cubic **K562** $= ps\mathcal{K}(X_5, X_{264}, X_3)$ with center X_5 passing through X_3, X_4, X_5,

- another cubic with concurring asymptotes for $\Omega = X_{140}$ passing through X_3, X_4, X_{140}, X_{276},

- the nodal cubic $ps\mathcal{K}(X_{1147}, X_{264}, X_3)$ with node X_3 passing through X_3, X_{24}, having perpendicular nodal tangents,

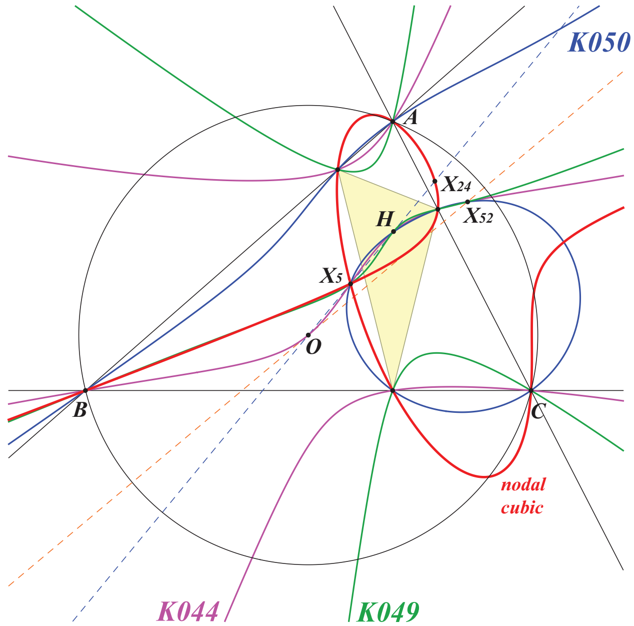

Figure 21: The net of cubics $ps\mathcal{K}(\Omega, X_4, X_5)$

- a pencil of pivotal cubics passing through X_{1993} when Ω lies on the line passing through X_2, X_{95}, X_{97}, X_{233}, X_{275}, X_{317}, X_{577}, X_{3087}. This pencil contains the Euler perspector cubic **K045** $= ps\mathcal{K}(X_2, X_{264}, X_3)$ passing through X_2, X_3, X_4, X_{69}, X_{254}, X_{264}, X_{1993},
- only one circular cubic **K563** $= ps\mathcal{K}(X_{186}, X_{264}, X_3)$ passing through X_3, X_4, X_{93}, X_{186}, X_{1154}, X_{1300}, X_{3043}.

Remark : any cubic of this type that contains H must have its pseudo-pole on the Euler line and then its contains this pseudo-pole.

4.2.3 Example 5 : $P = X_7$ and $M = X_1$, see figures 23 and 24

This is a very interesting example giving a lot of cubics and we shall have a closer look at it. Note that the Miquel associate of the Gergonne point X_7 is the incenter X_1 of ABC hence we have a simple example of net containing a pencil of circular cubics. These have their singular focus on the circle with center X_{1385} (the midpoint of X_1, X_3) passing through X_{214} (the midpoint of X_1, X_{100}) whose radius is that of the nine point circle. The envelope of the real asymptote is the Steiner deltoid of ABC.

Each $ps\mathcal{K}(\Omega, X_7, X_1)$ meets the line at infinity and the circumcircle of ABC at the same points as two isogonal $p\mathcal{K}$ whose respective pivots are :

– the anticomplement P_∞ of the barycentric product $\Omega \times X_{312}$,

– the reflection P_O of P_∞ in the barycentric quotient of $X_{56} \div S$, where S is the cevapoint of X_6 and Ω i.e. the trilinear pole of the polar of Ω in the circumcircle of ABC.

The cubic $ps\mathcal{K}(\Omega, X_7, X_1)$ is

- nodal with node X_1 if and only if $\Omega = X_{56}$,
- equilateral if and only if $\Omega = X_{1393}$,
- circular if and only if Ω lies on the trilinear polar of X_{57} passing through X_{513}, X_{663}, X_{855}, X_{1149}, X_{1279}, X_{1284}, X_{1319}, X_{1455}, X_{1456}, X_{1457}, X_{1458}, X_{1459}, X_{1463}, X_{1464}, X_{1769}, X_{2605},

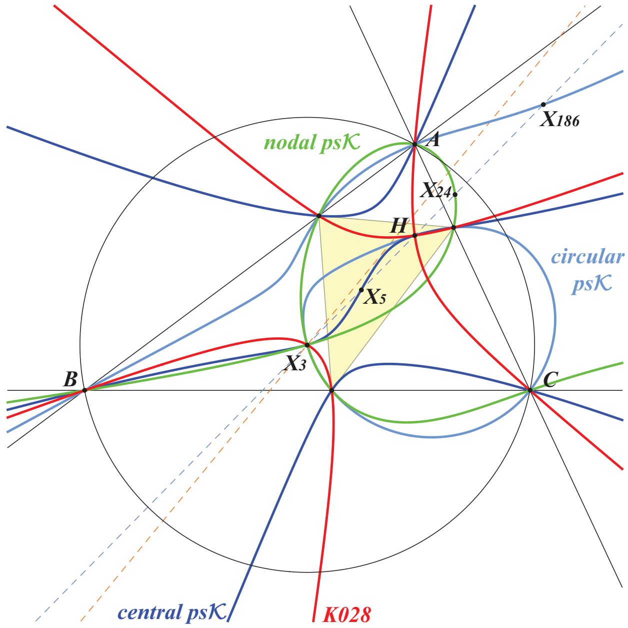

Figure 22: The net of cubics $ps\mathcal{K}(\Omega, X_{264}, X_3)$

- pivotal if and only if Ω lies on the trilinear polar of X_{934} passing through X_6, X_{57}, X_{222}, X_{223}, X_{269}, X_{610}, X_{910}, X_{1190}, X_{1407}, X_{1418}, X_{1419}, X_{1427}, X_{1439}, X_{1465}, X_{1615}, X_{1730}, X_{1763}, X_{2003}, X_{2097}, X_{2270}, X_{2999},
- central with center X_1 if and only if $\Omega = X_1$,
- a cubic $\mathcal{K}^+$ (with concurring asymptotes) if and only if Ω lies on a complicated cubic passing through X_1, X_{1394}.

These loci are represented on figure 23 and some special cubics are shown on figure 24. See annexe Table 5 for a list of special cubics with corresponding centers and Table 6 for other remarkable cubics.

The cubic **K561** $= ps\mathcal{K}(X_{603}, X_7, X_1)$ is particularly interesting since it has a lot of geometric properties. See figure 25.

1. it contains the incenter X_1 which is a flex on the cubic with an inflexional tangent passing through the orthocenter X_4. The harmonic polar of X_1 is the trilinear polar of X_{81} passing through X_{36}, X_{238}, X_{513}, X_{667}, X_{859}, X_{905}, X_{1019}, X_{1756}, X_{2530}, X_{3220}.
2. it also contains X_3, X_{56}, X_{224}, the infinite points of the orthocubic **K006**, the vertices P_a, P_b, P_c of the intouch triangle.
3. it meets the circumcircle again at the vertices A', B', C' of the circumcevian triangle of X_1. The tangents at these points concur at a point labelled X on the figure. X is not listed in the current version of [8] although it is the midpoint of X_1, X_{19} and the intersection of several lines such as X_1X_{19}, X_3X_{142}, X_7X_{3220}, $X_{25}X_{226}$, $X_{31}X_{57}$, etc.
4. the cubic meets the sidelines of triangle $A'B'C'$ at three other points A'', B'', C'' vertices of a triangle perspective with the three triangles ABC, $A'B'C'$, $P_aP_bP_c$ at X_3, X_{991}, X_1 respectively.
5. its pseudo-isopivot is X_{48} (on the line X_1X_{19}) and the cubic meets the circum-conic with perspector X_{48} again at three points Q_1, Q_2, Q_3 with tangents obviously passing through X_{48}.

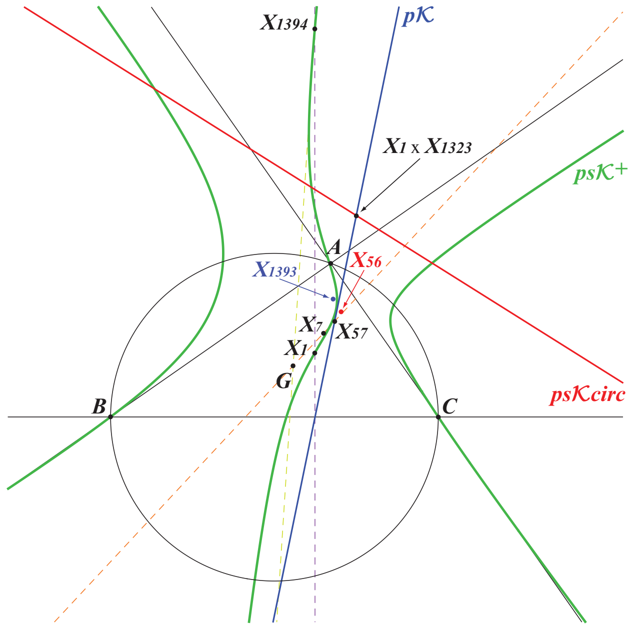

Figure 23: Loci related to cubics $ps\mathcal{K}(\Omega, X_7, X_1)$

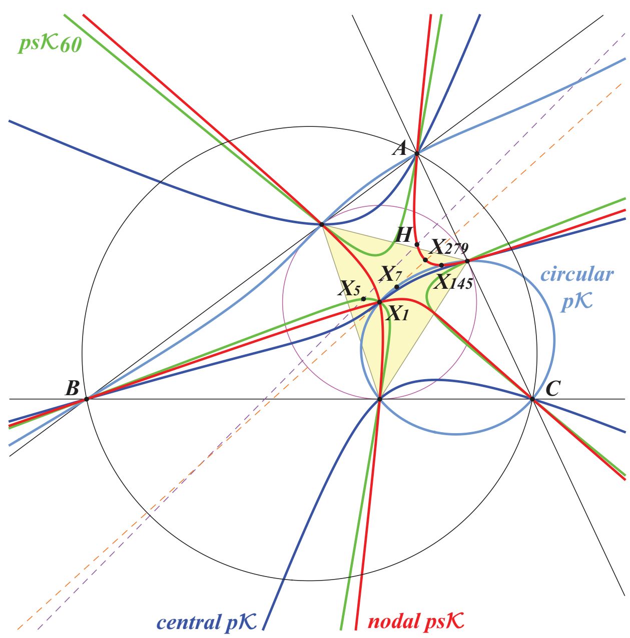

Figure 24: Examples of special cubics $ps\mathcal{K}(\Omega, X_7, X_1)$

6. it meets the incircle again at R_1, R_2, R_3 on the sidelines of triangle $Q_1Q_2Q_3$ and the triangles $Q_1Q_2Q_3$, $R_1R_2R_3$ are perspective at a point labelled Y on the figure.

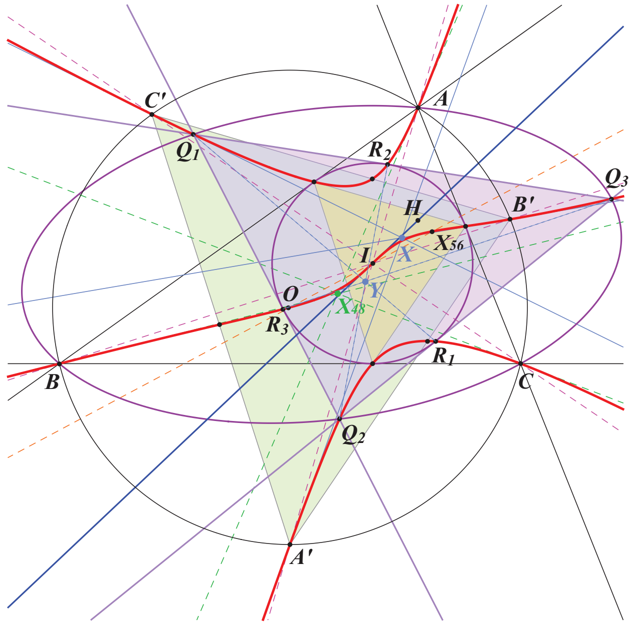

Figure 25: **K561** $= ps\mathcal{K}(X_{603}, X_7, X_1)$

5 Annexe

5.1 Table 3 : Circular cubics $ps\mathcal{K}$ with pseudo-pivot G

M is the real infinite point of the cubic which also contains the isogonal conjugate of M. The pseudo-pole Ω lies on the Lemoine axis. These cubics always contain the circumcenter $O = X_3$.

5.2 Table 4 : Other remarkable cubics $ps\mathcal{K}(\Omega, X_2, X_3)$

This table contains remarkable cubics that are not pivotal (obtained when Ω lies on the Brocard axis) nor circular (obtained when Ω lies on the Lemoine axis, as in Table 3).

5.3 Table 5 : Special cubics of the net $ps\mathcal{K}(\Omega, X_7, X_1)$

5.4 Table 6 : Other cubics of the net $ps\mathcal{K}(\Omega, X_7, X_1)$

note 1 : any cubic $ps\mathcal{K}(\Omega, X_7, X_1)$ passing through its pseudo-pole Ω also contains X_{145}.

note 2 : a cubic $ps\mathcal{K}(\Omega, X_7, X_1)$ presents a singularity at X_1 if and only if Ω lies on $ps\mathcal{K}(X_{1397}, X_{57}, X_1)$ passing through X_1, X_{41}, X_{56}, X_{208}, X_{603}. This singularity is a node when $\Omega = X_{56}$, a flex otherwise.

M	Ω	cubic	centers on the cubic
X_{30}	X_{1495}	**K446**	X_3, X_4, X_{30}, X_{74}, X_{133}, X_{1511}
X_{511}	X_{237}	**K570**	X_3, X_{32}, X_{98}, X_{132}, X_{511}
X_{517}			X_3, X_{56}, X_{104}, X_{517}, X_{1145}
X_{519}	X_{902}		X_1, X_3, X_{106}, X_{214}, X_{519}, X_{1319}
X_{520}			X_3, X_{107}, X_{125}, X_{520}
X_{521}			X_3, X_{11}, X_{108}, X_{521}
X_{523}	X_{512}	**K567**	X_3, X_{110}, X_{136}, X_{523}
X_{524}	X_{187}	**K043**	X_2, X_3, X_6, X_{67}, X_{111}, X_{187}, X_{468}, X_{524}, X_{1560}, X_{2482}
X_{525}	X_{647}		X_3, X_{112}, X_{115}, X_{525}
X_{527}	X_{1055}		X_3, X_9, X_{57}, X_{527}, X_{1155}, X_{2291}
X_{532}			X_3, X_{13}, X_{16}, X_{532}, X_{618}, X_{2380}
X_{533}			X_3, X_{14}, X_{15}, X_{533}, X_{619}, X_{2381}
X_{539}			X_3, X_{128}, X_{539}, X_{2383}
X_{732}			X_3, X_{39}, X_{732}, X_{733}, X_{1691}
X_{758}			X_3, X_{10}, X_{36}, X_{65}, X_{758}, X_{759}
X_{912}			X_3, X_{119}, X_{912}, X_{915}
X_{916}			X_3, X_{118}, X_{916}, X_{917}
X_{924}			X_3, X_{135}, X_{924}, X_{925}
X_{1154}			X_3, X_5, X_{186}, X_{1141}, X_{1154}
X_{2393}			X_3, X_{25}, X_{206}, X_{858}, X_{2373}, X_{2393}
X_{2574}			X_3, X_{1113}, X_{1313}, X_{2574}
X_{2575}			X_3, X_{1114}, X_{1312}, X_{2575}

Table 3: Circular cubics $ps\mathcal{K}$ with pseudo-pivot G

Ω	cubic $ps\mathcal{K}(\Omega, X_2, X_3)$ and/or centers	comments
X_2	**K512** / X_3, X_{76}, X_{3224}	
X_{25}	**K376** / X_3, X_4, X_{64}	
X_{51}	**K026** / X_3, X_4, X_5	**equilateral, central**
X_{55}	**K259** / X_1, X_3, X_8, X_{220}, X_{277}, X_{3160}	**nodal**
X_{110}	**K559** / X_3, X_{99}, X_{1379}, X_{1380}, X_{2479}, X_{2480}	
X_{154}	**K426** / X_3, X_4, X_{20}	**central** $ps\mathcal{K}^+$
X_{184}	**K009** / X_3, X_4, X_{32}, X_{56}, X_{1147}	**nodal**
X_{13366}	**K569** / X_3, X_4, X_{54}, X_{140}, X_{1493}	

Table 4: Special cubics of the net $ps\mathcal{K}(\Omega, X_2, X_3)$

Ω	cubic $ps\mathcal{K}(\Omega, X_7, X_1)$ and/or centers	comments
X_1	**K830** / X_1, X_8, X_{145}, X_{2137}	**central**
X_6	X_1, X_7, X_9, X_{55}, X_{57}, X_{218}, X_{277}, X_{3174}	$p\mathcal{K}$
X_{56}	**K360** / X_1, X_4, X_{56}, X_{145}, X_{218}, X_{279}, X_{1433}	**nodal**
X_{57}	**K365** / X_1, X_2, X_7, X_{57}, X_{145}, X_{174}, X_{1488}, X_{2089}	$p\mathcal{K}$
X_{222}	X_1, X_3, X_7, X_{57}, X_{63}, X_{77}, X_{90}, X_{224}, X_{3173}	$p\mathcal{K}$
X_{223}	**K333** / X_1, X_7, X_{40}, X_{57}, X_{329}, X_{347}	$p\mathcal{K}$
X_{269}	X_1, X_7, X_{57}, X_{279}, X_{1659}, X_{3062}	$p\mathcal{K}$
X_{513}	X_1, X_{11}, X_{108}, X_{522}	circular
X_{610}	X_1, X_7, X_{20}, X_{57}	$p\mathcal{K}$
X_{663}	X_1, X_{3022}	circular
X_{910}	X_1, X_7, X_{57}, X_{294}, X_{516}	$p\mathcal{K}$
X_{1254}	**K1058** / X_1, X_{12}, X_{65}, X_{85}	**nodal**
X_{1279}	X_1, X_{3021}	circular
X_{1284}	X_1, X_{740}, X_{3027}	circular
X_{1319}	X_1, X_{80}, X_{145}, X_{519}, X_{1317}, X_{1319}	circular
X_{1393}	X_1, X_5	**equilateral**
X_{1394}	X_1, X_{20}, X_{279}	$ps\mathcal{K}^+$
X_{1407}	X_1, X_7, X_{56}, X_{57}, X_{84}, X_{222}, X_{269}, X_{278}	$p\mathcal{K}$
X_{1418}	X_1, X_7, X_{57}, X_{142}, X_{354}	$p\mathcal{K}$
X_{1419}	X_1, X_7, X_{57}, X_{144}, X_{165}, X_{3160}	$p\mathcal{K}$
X_{1427}	**K964** / X_1, X_4, X_7, X_{57}, X_{65}, X_{196}, X_{226}, X_{1439}	$p\mathcal{K}$
X_{1439}	X_1, X_7, X_{57}, X_{307}, X_{1214}	$p\mathcal{K}$
X_{1455}	X_1, X_4, X_{104}, X_{515}, X_{1359}	circular
X_{1456}	X_1, X_{105}, X_{279}, X_{516}, X_{1360}	circular
X_{1457}	X_1, X_{56}, X_{106}, X_{517}, X_{1361}, X_{1785}	circular
X_{1458}	X_1, X_{218}, X_{518}, X_{1362}, X_{1477}	circular
X_{1459}	X_1, X_{521}, X_{1364}	circular
X_{1463}	X_1, X_{726}	circular
X_{1464}	X_1, X_{36}, X_{65}, X_{758}, X_{3028}	circular
X_{1465}	X_1, X_7, X_{57}, X_{80}, X_{88}, X_{517}, X_{908}	$p\mathcal{K}$
X_{1615}	X_1, X_7, X_{57}, X_{2951}	$p\mathcal{K}$
X_{1769}	X_1, X_{2804}	circular
X_{2003}	X_1, X_7, X_{35}, X_{57}, X_{1442}, X_{3219}	$p\mathcal{K}$
X_{2097}	X_1, X_7, X_{57}	$p\mathcal{K}$
X_{2270}	X_1, X_7, X_{57}, X_{962}	$p\mathcal{K}$
X_{2605}	X_1, X_{3024}	circular
X_{2999}	X_1, X_7, X_{57}, X_{1697}	$p\mathcal{K}$
X_{6610}	**K949** / X_1, X_7, X_{57}, X_{527}, X_{1155}, X_{1156}, X_{1323}	circular $p\mathcal{K}$

Table 5: Special cubics of the net $ps\mathcal{K}(\Omega, X_7, X_1)$

Ω	cubic $ps\mathcal{K}(\Omega, X_7, X_1)$ and/or centers	comments
X_3	X_1, X_3, X_{78}, X_{145}, X_{224}	note 1
X_{31}	X_1, X_{33}, X_{55}, X_{56}, X_{1486}, X_{2191}, X_{3174}	
X_{41}	X_1, X_{218}, X_{220}	note 2
X_{55}	X_1, X_{55}, X_{145}, X_{200}, X_{3174}	note 1
X_{65}	**K571** / X_1, X_{10}, X_{65}, X_{145}	see §2.4
X_{73}	X_1, X_3, X_{65}, X_{72}, X_{218}, X_{224}	
X_{208}	X_1, X_4	note 2
X_{212}	X_1, X_3, X_{55}, X_{224}, X_{1260}, X_{3174}	
X_{603}	**K561** / X_1, X_3, X_{56}, X_{224}	note 2
X_{604}	X_1, X_6, X_{218}, X_{222}, X_{269}, X_{2192}	
X_{1214}	X_1, X_{145}, X_{306}, X_{1214}, X_{2997}	note 1
X_{1400}	X_1, X_{33}, X_{37}, X_{218}, X_{226}, X_{1486}	
X_{1460}	X_1, X_{33}, X_{145}, X_{388}, X_{612}, X_{1460}, X_{1486}	note 1
X_{6611}	**K713** /X_1, X_4, X_{222}, X_{223}, X_{269}, X_{347}	

Table 6: Other cubics of the net $ps\mathcal{K}(\Omega, X_7, X_1)$

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