Cubics related with Parallelogic Pedal Triangles

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Abstract

We study the parallelogic pedal triangles of two points and several related cubic curves.

1 Introduction

Consider two points $P = p : q : r$ and $Q = u : v : w$ in barycentric coordinates with respect to the usual reference triangle ABC. Let $P_aP_bP_c$ and $Q_aQ_bQ_c$ be their respective pedal triangles.

These triangles are said to be parallelogic when the parallels drawn through the vertices of one of them to the sidelines of the other are concurrent at a point called center of parallelogy.

There are obviously two (not necessarily distinct, see below) centers of parallelogy denoted P_1 and P_2.

More precisely, the parallels $\mathcal{L}_a$ (resp. $\mathcal{L}_b$, resp. $\mathcal{L}_c$) at P_a (resp. P_b, resp. P_c) to the sidelines Q_bQ_c (resp. Q_cQ_a, resp. Q_aQ_b) concur (at P_1) if and only if

$$(p+q+r)^2 \left(a^2vw+b^2uw+c^2uv\right) \left(a^2qw+a^2rv+b^2pw+b^2ru+c^2pv+c^2qu\right) = 0. \quad (1)$$

The corresponding equation for P_2 is easily obtained by swapping the coordinates of P and Q in equation (1).

This shows that one of the three following conditions must be realized :

1. P (or Q) is a point at infinity,
2. P (or Q) is a point on the circumcircle $(\mathcal{O})$ of ABC,
3. P and Q are conjugated in $(\mathcal{O})$.

The first two correspond to an improper pedal triangle consisting in either the line at infinity or a Simson line. On the other hand, if $P = O$, its polar in $(\mathcal{O})$ is the line at infinity. These have to be discarded and will be treated as limit cases in the sequel. Finally we have

Theorem 1. *For any finite point $P \neq O$ not lying on $(\mathcal{O})$, the pedal triangles of P and Q are parallelogic if and only if Q lies on the polar $\mathcal{L}_P$ of P in $(\mathcal{O})$.*

A special case

When Q is the inverse Q_o of P in $(\mathcal{O})$ hence the intersection of $\mathcal{L}_P$ and the line OP, the two triangles are also orthologic and indirectly similar. Moreover, the two corresponding points P_1 and P_2 coincide.

2 The centers of parallelogy

Let now P be a fixed point following the conditions of Theorem 1. When Q traverses $\mathcal{L}_P$,

- the locus $(\mathcal{P}_1)$ of P_1 is a conic passing through P_a, P_b, P_c,
- the locus $(\mathcal{P}_2)$ of P_2 is a line,
- the envelope $(\mathcal{P}_{12})$ of P_1P_2 is a conic inscribed in ABC. See figure 1.

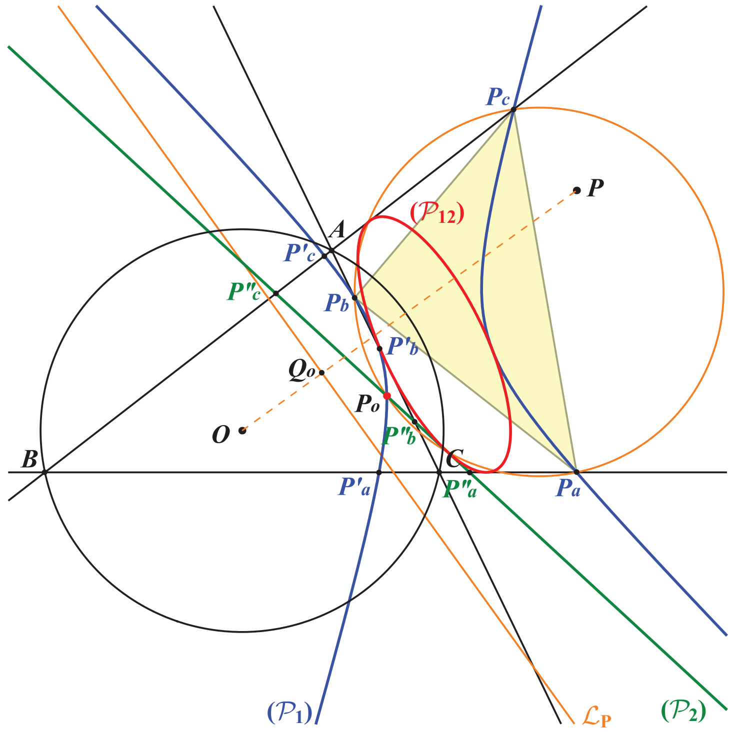

Figure 1: The loci $(\mathcal{P}_1)$, $(\mathcal{P}_2)$, $(\mathcal{P}_{12})$

2.1 Properties of $(\mathcal{P}_1)$

An easy computation shows that the center Ω_1 of $(\mathcal{P}_1)$ is the homothetic of the centroid G_P of $P_aP_bP_c$ under $h(P, 3/2)$. We have $G_P = a^2(2b^2c^2p + c^2S_Cq + b^2S_Br) \; : \; :$ and then $\Omega_1 = a^2(b^2c^2p + c^2S_Cq + b^2S_Br) \; : \; : \; .$ This gives an easy construction of $(\mathcal{P}_1)$. Note that the reflections of P_a, P_b, P_c about Ω_1 are obtained when Q is a common point of $\mathcal{L}_P$ with the corresponding sideline of ABC.

When Q is the intersection of $\mathcal{L}_P$ and the line AO, the corresponding point P_1 is the second intersection of $(\mathcal{P}_1)$ with BC. This is the point $P'_a = 0 : c^2S_C\,(b^2\,p + S_C\,q) : b^2S_B(c^2\,p + S_B\,r)$. Similarly, two other points P'_b, P'_c on CA, AB are constructed.

Another interesting point P_o on $(\mathcal{P}_1)$ is obtained when Q is the point Q_o defined above. It appears that P_o also lies on $(\mathcal{P}_2)$ and on the pedal circle of P i.e. on the circumcircle of $P_aP_bP_c$.

The coordinates are : $P_o = \dfrac{(b^2 - c^2)p + a^2(q - r)}{c^2S_C\,q - b^2S_B\,r} \; : \; :$ hence P_o is the barycentric quotient of the infinite point of $\mathcal{L}_P$ and the trilinear pole of the line OP.

The limit cases (see figure 2) :

1. When P approaches O, $(\mathcal{P}_1)$ approaches the nine point circle,
2. When P approaches a point on $(\mathcal{O})$, $(\mathcal{P}_1)$ approaches the union of its Simson and Steiner lines,
3. When P approaches a point on the line at infinity, $(\mathcal{P}_1)$ approaches the union of the line at infinity and a tangent T_P to the MacBeath inconic. This tangent passes through the intersection E of the Simson lines of the intersections of the line OP with $(\mathcal{O})$.

Furthermore, when P is inside (resp. outside) $(\mathcal{O})$, the conic $(\mathcal{P}_1)$ is an ellipse (resp. a hyperbola).

Obviously, $(\mathcal{P}_1)$ is a bicevian conic when $P_aP_bP_c$ is a cevian triangle. This occurs when P lies on the Darboux cubic K004 but also when P lies on a cevian line of X_{64}.

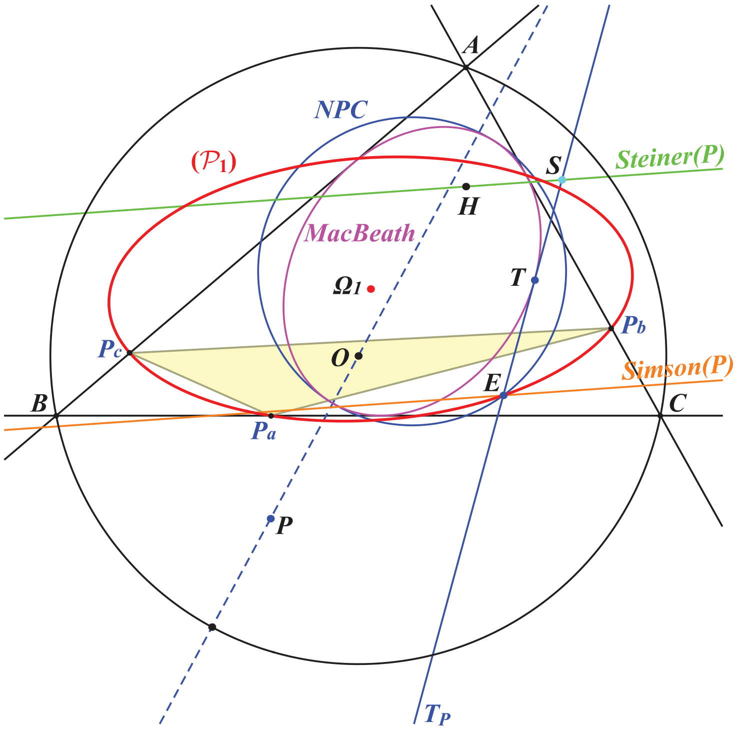

Figure 2: The locus $(\mathcal{P}_1)$ and its limit cases

When the line OP rotates about O, T_P and the Steiner line meet at S which lies on a nodal cubic with node H, passing through the vertices of the cevian triangles of H and X_{92} and also the traces of the trilinear polar of X_{92} on the sidelines of ABC. This cubic is the barycentric quotient of the cubic K215 by X_3. Recall that K215 is the locus of the common point N of the trilinear polars of M and its isogonal conjugate $M*$ when M traverses $(\mathcal{O})$. At last, note that this cubic is tritangent to the MacBeath inconic, the contacts being on the parallels at H to the asymptotes of the McCay cubic K003. See figure 3.

2.2 Properties of $(\mathcal{P}_2)$

$(\mathcal{P}_2)$ is the trilinear polar of $T_2 = \dfrac{c^2 S_C\, q - b^2 S_B\, r}{c^2\, q + b^2\, r} : : $, the barycentric quotient of the cevapoint of (K, P) and the trilinear pole of the line OP. It meets the sidelines of ABC at P''_a, P''_b, P''_c which correspond to the positions of P_1 at P_a, P_b, P_c.

Recall that $(\mathcal{P}_2)$ contains P_o obtained when $Q = Q_o$.

The limit cases :

1. When P approaches O, $(\mathcal{P}_2)$ approaches a tangent to the MacBeath inconic,
2. When P approaches a point on $(\mathcal{O})$, $(\mathcal{P}_2)$ approaches its Simson line,
3. When P approaches a point on the line at infinity, $(\mathcal{P}_2)$ approaches the reflection of the tangent above about the Simson line above.

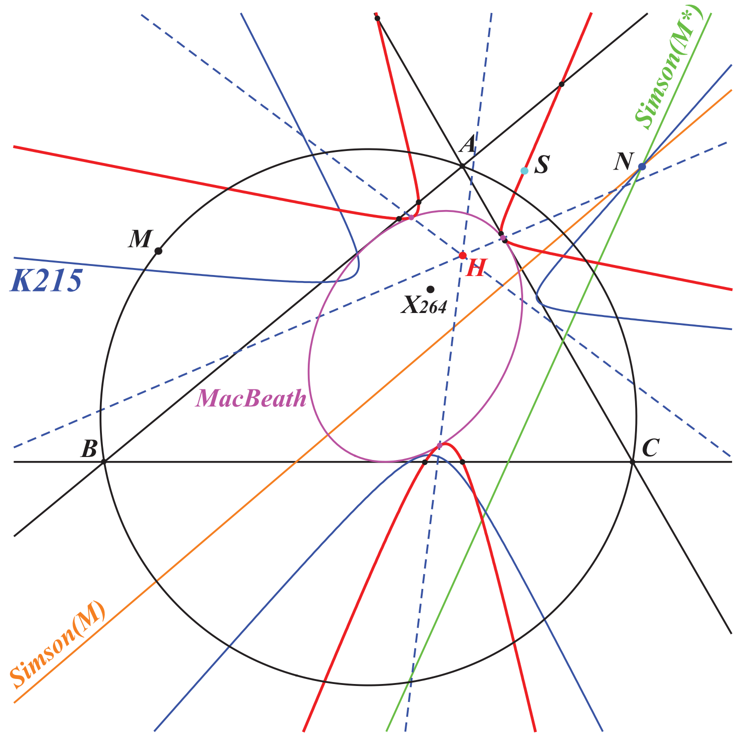

Figure 3:

2.3 Properties of $(\mathcal{P}_{12})$

$(\mathcal{P}_{12})$ is the inconic with perspector $T_{12} = \dfrac{1}{a^2 S_A\,(c^2\,q + b^2\,r)}$: : , the barycentric quotient of the cevapoint of (K, P) and O.

Its center is Ω_1 as above. In other words, $(\mathcal{P}_1)$ and $(\mathcal{P}_{12})$ are concentric.

Moreover, $(\mathcal{P}_{12})$ is tangent to $(\mathcal{P}_2)$ and bitangent to $(\mathcal{P}_1)$.

The limit cases :

1. When P approaches O, $(\mathcal{P}_{12})$ approaches the MacBeath inconic,
2. When P approaches a point on $(\mathcal{O})$, $(\mathcal{P}_{12})$ approaches an inscribed conic passing through H and tangent at H to its Steiner line,
3. When P approaches a point on the line at infinity, $(\mathcal{P}_{12})$ approaches an inscribed parabola.

3 Cubics $\mathcal{K}_2(M)$ related with $(\mathcal{P}_2)$

The line $(\mathcal{P}_2)$ passes through a fixed point $M = u : v : w$ if and only if P lies on the circum-cubic $\mathcal{K}_2(M)$ with equation :

$$\sum_{\text{cyclic}} u\,\frac{c^2y + b^2z}{c^2S_C\,y - b^2S_B\,z} = 0 \text{ or } \sum_{\text{cyclic}} u\,(c^2y+b^2z)(a^2S_A\,z - c^2S_C\,x)(b^2S_B\,x - a^2S_A\,y) = 0 \quad (2)$$

$\mathcal{K}_2(M)$ is a nodal cubic with node O and all these cubics form a net.

Note that $\mathcal{K}_2(A)$ decomposes into the tangent at A to $(\mathcal{O})$ and the cevian lines of O passing through B, C.

It follows that any cubic $\mathcal{K}_2(M)$ of the net may be written under the form :

$$\mathcal{K}_2(M) = u\,\mathcal{K}_2(A) + v\,\mathcal{K}_2(B) + w\,\mathcal{K}_2(C) \tag{3}$$

Special cubics :

This net contains many interesting cubics of different kinds but no proper $p\mathcal{K}$ since an undecomposed $p\mathcal{K}$ cannot be nodal.

1. When M lies at infinity, $\mathcal{K}_2(M)$ is a circular cubic with singular focus on the Kosnita circle i.e. the circumcircle of the Kosnita triangle. This circle is centered at X_{1658} and passes through X_{5961} corresponding to the Jerabek strophoid K039. Note that $\mathcal{K}_2(M)$ decomposes when M is a vertex of the Kosnita triangle.

2. When M lies on the nine point circle, $\mathcal{K}_2(M)$ splits into a line and a circum-conic, both passing through O and meeting again at a point S on the Lemoine cubic K009.

   The line is the parallel at O to the Simson line of the anticomplement of M. The point S also lies on the line HM.

3. When M lies on the Euler line, $\mathcal{K}_2(M)$ is $c\mathcal{K}$ with singular point O and root on the line GK. For example, $\mathcal{K}_2(O)$ is K658 $= c\mathcal{K}(\#X_3, X_{1993})$. The only $c\mathcal{K}_0$ is $\mathcal{K}_2(X_{297})$.

4. The net contains only one equilateral cubic namely $\mathcal{K}_2(X_{5449})$ with asymptotes parallel to those of K024.

5. When M lies on the trilinear polar of X_{5392} – which is the directrix associated with O in the MacBeath inconic – the nodal tangents at O are perpendicular. This is the case of $\mathcal{K}_2(X_{5961})$ = K039.

6. When M lies on the MacBeath inconic, $\mathcal{K}_2(M)$ is a cuspidal cubic with cusp obviously O. The cuspidal tangents of $\mathcal{K}_2(X_{2972})$, $\mathcal{K}_2(X_{339})$ are the Euler line and the Brocard axis respectively.

4 Envelope of the line QP_2

4.1 Properties

When Q traverses $\mathcal{L}_P$, the line QP_2 envelopes a parabola $\mathcal{P}_P$ tangent to $\mathcal{L}_P$ and $(\mathcal{P}_2)$.

The directrix Δ_P is : $\displaystyle\sum_{\text{cyclic}} b^2c^2p\left(b^2r+c^2q\right)\left(a^2q-a^2r+b^2p-c^2p\right)x = 0.$

The focus F_P is a point on the circumcircle $(\mathcal{O})$ which is the barycentric product of the isogonal conjugate of Q_o and the cevapoint of (K, P).

Note that Δ_P is perpendicular to the Simson line of F_P. See figure 4.

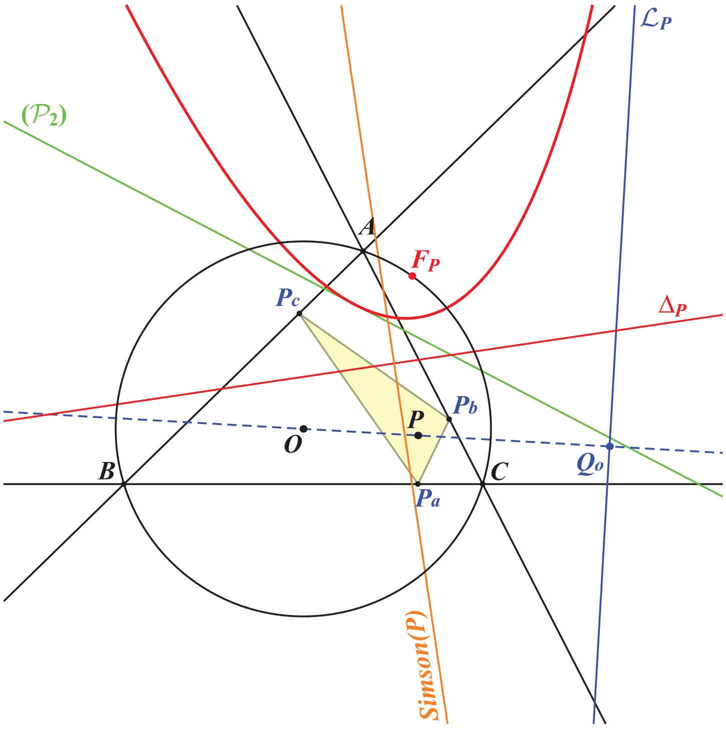

Figure 4: Parabola envelope of the line QP_2

4.2 Cubics related with F_P

For a given point F on $(\mathcal{O})$, there are infinitely many parabolas $\mathcal{P}_P$ with focus F and their directrices are all parallel. The corresponding points P lie on a circular circum-cubic passing through O and the vertices of the tangential triangle. All these cubics are in a same pencil and their singular foci lie on the Kosnita circle.

When $F = X_{111}$, we find the only $p\mathcal{K}$ of the pencil namely K108 $= p\mathcal{K}(X_{32}, X_{23})$. There are two other $ps\mathcal{K}$s in the pencil obtained when F is on the orthic axis but they real only if ABC is obtuse angle.

With $F = X_{74}$, we find an interesting cubic which is tangent at O to the Euler line and which passes also through X_{64}, X_{74}, X_{186}, X_{399} and X_{6000} on the line at infinity. It has nine identified common points with the Neuberg cubic K001 since the tangents at X_{74} are the same. It is a $ps\mathcal{K}$ with respect to the tangential triangle with pseudo-pivot K and pseudo-isopivot the intersection of the lines X_3, X_{66} and X_{23}, X_{385}. See figure 5.

4.3 Cubics related with Δ_P

The form of the equation of Δ_P naturally leads to another net of cubics.

Indeed, each cubic $\mathcal{K}_\Delta(M)$ is the locus of P such that Δ_P passes through a fixed point $M = u : v : w$. This generalizes the previous paragraph where M was a point on the line at infinity.

For any M, $\mathcal{K}_\Delta(M)$ is a circum-cubic passing through O and the vertices K_a, K_b, K_c of the tangential triangle. Its equation is easily obtained from that of Δ_P :

$$\sum_{\text{cyclic}} b^2c^2\, u\, x\, (b^2z + c^2y)\, \left[a^2(y-z) + (b^2-c^2)\, x\right] = 0.$$

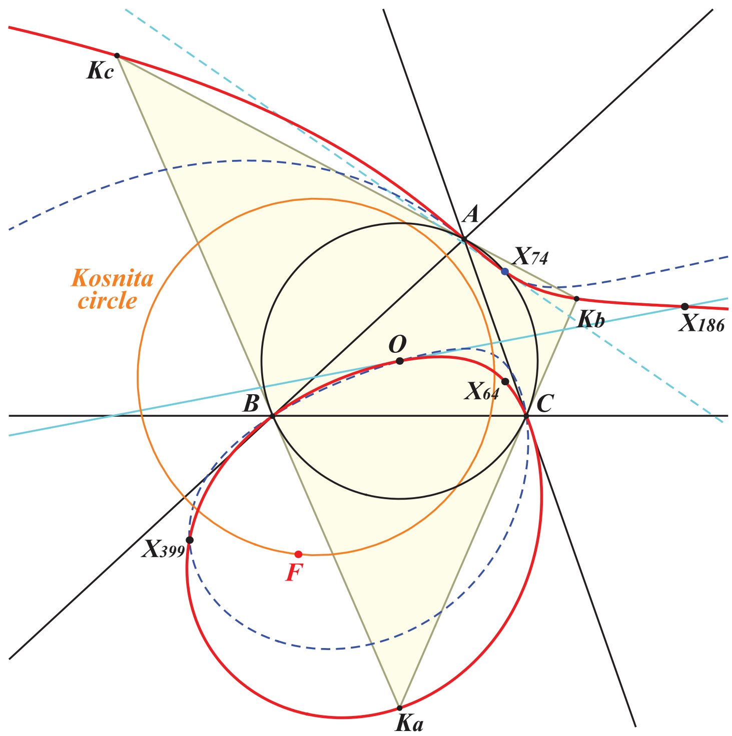

Figure 5: Cubic related with $F_P = X_{74}$

The tangents at K_a, K_b, K_c to $\mathcal{K}_\Delta(M)$ concur at X which is the Ceva-conjugate of tcM and X_6, where tcM is the isotomic conjugate of the complement of M. From this, we obtain

Theorem 2. *$\mathcal{K}_\Delta(M)$ is a $ps\mathcal{K}$ with pseudo-pivot X_6 and pseudo-isopivot $X = tcM/X_6$ with respect to the tangential triangle.*

In general, X does not lie on $\mathcal{K}_\Delta(M)$ unless M lie on a sideline of the antimedial triangle (giving a decomposed cubic) or on the line GK (giving a $p\mathcal{K}$ with respect to both triangles ABC and $K_aK_bK_c$). See below.

$\mathcal{K}_\Delta(M)$ always contains the isogonal conjugate M^* of M but generally not M unless it lies on $(\mathcal{O})$ or on the Stammler hyperbola. In the former case, $\mathcal{K}_\Delta(M)$ meets $(\mathcal{O})$ again at two points lying on the parallel at O to the Simson line of M.

$\mathcal{K}_\Delta(M)$ meets the circumcircle at the same points as $p\mathcal{K}(X_6, M)$. It follows that these two cubics meet again at three points on a line passing through M^*. This line is the trilinear polar of the barycentric quotient $X_{110} \div M$.

$\mathcal{K}_\Delta(M)$ meets the line at infinity at the same points as the isogonal $p\mathcal{K}$ with pivot P_M, anticomplement of the isogonal conjugate of the cevapoint of (O, P). This point is $P_M = a^2[b^2c^2\,u + c^2(c^2-a^2)\,v + b^2(b^2-a^2)\,w] \; : : .$ The six remaining common (finite) points lie on the circum-conic with perspector $X_{523} \times M$.

For instance, with $M = O$, $\mathcal{K}_\Delta(M)$ is K405 meeting the line at infinity at the same points as the Darboux cubic K004, the remaining common points being on the Jerabek hyperbola. K405 meets $(\mathcal{O})$ at the same points as the McCay cubic K003 i.e. at the vertices of the circum-normal triangle and the remaining common points lie on the Euler line, namely O and H.

The net of cubics $\mathcal{K}_\Delta(M)$ contains a lot of remarkable special cubics. See figure 6.

1. $\mathcal{K}_\Delta(M)$ is a $\mathcal{K}_0$ if and only if M lies on the line GK. In this case, for any M on GK, $\mathcal{K}_\Delta(M)$ is a $p\mathcal{K}$ with pole X_{32} and pivot P_M on the Euler line. The isopivot lies on the circum-conic with perspector X_{3049}.

   Recall that $\mathcal{K}_\Delta(M)$ is also the $p\mathcal{K}$ with pivot X_6 and isopivot X as above with respect to the tangential triangle.

   These cubics form a pencil and also contain X_6 and X_{25}. The line M, P_M is tangent to the Stammler hyperbola and the point of tangency N lies on the cubic since it is the P_M-Ceva conjugate of O.

2. $\mathcal{K}_\Delta(M)$ is a $ps\mathcal{K}$ if and only if M lies on the circum-conic with perspector O and center K. Two of these cubics are in fact $p\mathcal{K}$s since the line GK meets the circum-conic at two real points. In every case,

   - the pseudo-pole lies on $ps\mathcal{K}(X_{1501} \times X_3, X_6, X_3)$, a nodal cubic with node X_{32} passing through O, X_{3172},
   - the pseudo-pivot lies on $(\mathcal{O})$,
   - the pseudo-isopivot lies on the Lemoine axis.

3. $\mathcal{K}_\Delta(M)$ is an equilateral cubic if and only if $P_M = O$ hence $M = X_{1147}$. This gives a cubic passing through X_3, X_{156}, X_{847}, with three real asymptotes parallel to those of the McCay cubic K003.

4. As already said in §4.2, $\mathcal{K}_\Delta(M)$ is a circular cubic if and only if P_M and therefore M lies on the line at infinity. This gives another pencil of cubics with singular foci on the Kosnita circle.

5. $\mathcal{K}_\Delta(M)$ is a singular cubic if and only if M lies on a sideline of the antimedial triangle (giving a decomposed cubic) or on a complicated circum-sextic passing through H, X_{145} and meeting the line GK at three double points. The tangents at A, B, C are the symmedians. This sextic is bitangent to each sideline of the antimedial triangle and meets these sidelines again at three points on the circum-conic with center O. For example, $\mathcal{K}_\Delta(H)$ is K389.

6. $\mathcal{K}_\Delta(M)$ has concurring asymptotes if and only if M lies on a cubic passing through X_{20}, X_{54}, X_{155}. These three points correspond to three central cubics with centers O, X_5, X_{1147} respectively.

7. The polar conic of O in $\mathcal{K}_\Delta(M)$ is degenerated (i.e. O is a node or an inflection point) if and only if M lies on the cubic K617 passing through X_4, X_{20}, X_{68}, X_{254}, X_{315}, X_{2996}.

8. For M not lying on the line GK nor on the circum-conic with center O, $\mathcal{K}_\Delta(M)$ is a $n\mathcal{K}$ if and only if M lies on a cubic $ps\mathcal{K}(X_{112}, X_{648}, ?)$ having the same infinite points as $n\mathcal{K}_0(X_6, X_{1368})$ and meeting the circumcircle at the same points as $n\mathcal{K}_0(X_6, X_{22})$.

4.4 Other cubics related with $\mathcal{K}_\Delta(M)$

The complement of the isogonal conjugate of $\mathcal{K}_\Delta(M)$ is $ps\mathcal{K}(X_6 \times cM, X_2, X_3)$ where cM is the complement of M. This latter cubic is a circum-cubic of ABC passing through O and the midpoints of ABC. The tangents at A, B, C concur at $X_6 \times cM$.

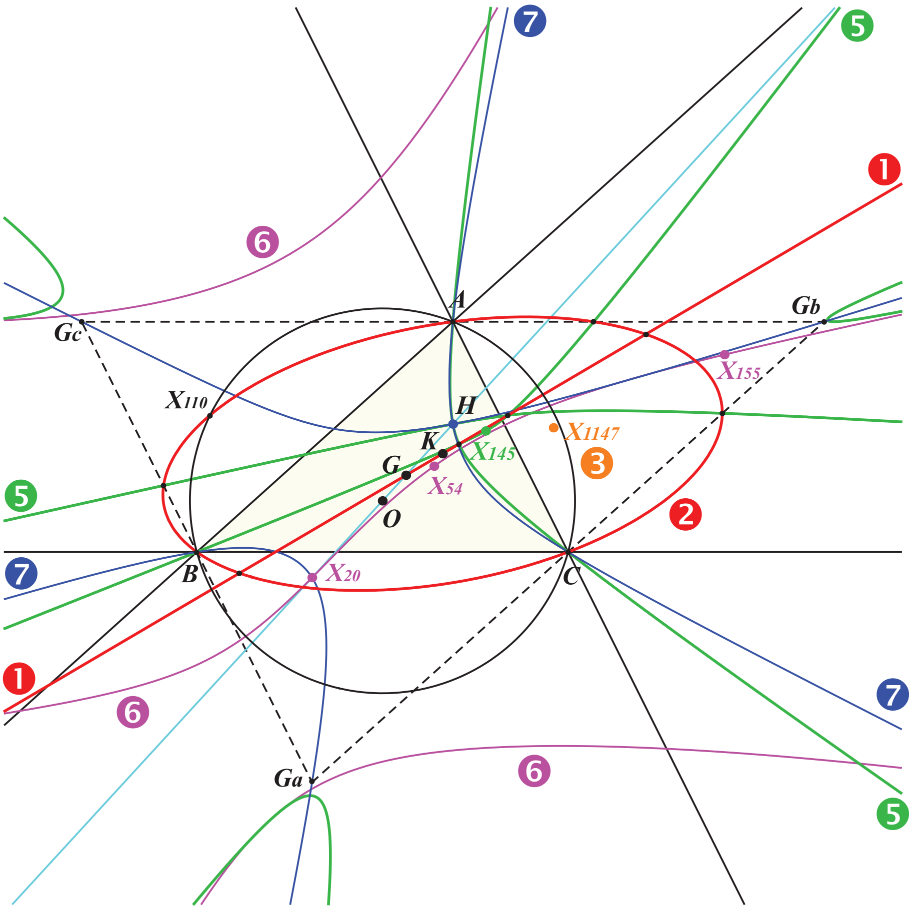

Figure 6: Loci related with special cubics $\mathcal{K}_{\Delta}(M)$

4.5 Table of cubics $\mathcal{K}_{\Delta}(M)$

The following Table 1 gives a selection of $\mathcal{K}_{\Delta}(M)$.
Notes :

- $M_{431} = a(a+b)(a+c)(a^3 - a^2b - a^2c - ab^2 - 2abc - ac^2 + b^3 + b^2c + bc^2 + c^3)$,
  SEARCH = -5.21240908380449.
- $M_{478} = a^2(a^6 - a^4b^2 - a^2b^4 + b^6 - a^4c^2 + 5a^2b^2c^2 - 2b^4c^2 - a^2c^4 - 2b^2c^4 + c^6)$,
  SEARCH = -0.299019495272116.

Table 1: Remarkable cubics $\mathcal{K}_\Delta(M)$

M	cubic / centers	type	remarks
X_1	X_1, X_3, X_{56}, X_{2217}		
X_2	K172	$p\mathcal{K}$	
X_3	K405		
X_4	K389		nodal
X_5	X_3, X_{54}, X_{64}, X_{195}		
X_6	K177	$p\mathcal{K}$	
X_{20}	X_3, X_{64}, X_{1498}		central
X_{30}	X_3, X_{64}, X_{74}, X_{186}, X_{399}, X_{6000}, see §4.2		circular
X_{54}	X_3, X_4, X_5		central
X_{69}	K174	$p\mathcal{K}$	
X_{81}	K430	$p\mathcal{K}$	
X_{99}	X_3, X_{99}, X_{512}, X_{1379}, X_{1380}, X_{3224}		
X_{100}	X_3, X_{100}, X_{513}, X_{1381}, X_{1382}		
X_{110}	X_3, X_{110}, X_{523}, X_{1113}, X_{1114}, X_{2574}, X_{2575}	$ps\mathcal{K}$	
X_{141}	X_3, X_6, X_{25}, X_{39}, X_{251}, X_{2916}	$p\mathcal{K}$	
X_{145}	X_3, X_{56}, X_{221}, X_{3207}		nodal
X_{155}	X_3, X_{155}, X_{254}, X_{1147}		central
X_{193}	K171	$p\mathcal{K}$	
X_{323}	K495	$p\mathcal{K}$	
X_{385}	X_3, X_6, X_{25}, X_{98}, X_{237}, X_{694}, X_{1691}, X_{1971}, X_{1987}	$p\mathcal{K}$	
X_{394}	K236	$p\mathcal{K}$	
X_{476}	X_3, X_{74}, X_{110}, X_{186}, X_{476}, X_{526}		
X_{511}	X_3, X_{98}, X_{1503}, X_{1676}, X_{1677}, X_{1691}, X_{2353}		circular
X_{524}	K108	$p\mathcal{K}$	circular
X_{527}	X_3, X_{55}, X_{1436}, X_{2078}, X_{2291}		circular
X_{758}	X_1, X_3, X_{759}, X_{1324}, X_{2217}, X_{2948}		circular
X_{805}	X_3, X_{98}, X_{99}, X_{804}, X_{805}, X_{1691}		
X_{895}	X_3, X_{23}, X_{67}, X_{468}, X_{1177}	$ps\mathcal{K}$	
X_{901}	X_3, X_{100}, X_{104}, X_{900}, X_{901}, X_{1319}		
X_{927}	X_3, X_{101}, X_{103}, X_{926}, X_{927}		
X_{1147}	X_3, X_{156}, X_{847}		equilateral
X_{1992}	X_3, X_6, X_{25}, X_{1384}, X_{1995}	$p\mathcal{K}$	
X_{1993}	K176	$p\mathcal{K}$	
X_{1994}	X_3, X_5, X_6, X_{25}, X_{195}, X_{2963}, X_{2965}	$p\mathcal{K}$	
X_{2287}	X_3, X_6, X_{25}, X_{610}, X_{1427}, X_{1817}, X_{2155}	$p\mathcal{K}$	
X_{2393}	X_3, X_{22}, X_{67}, X_{468}, X_{2373}, X_{2936}		circular
X_{3580}	X_3, X_6, X_{25}, X_{74}, X_{186}, X_{1495}, X_{2931}, X_{3003}	$p\mathcal{K}$	
M_{431}	K431	$p\mathcal{K}$	
M_{478}	K478	$p\mathcal{K}$	

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