Inscribed Nephroids and Related Curves

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Abstract

We study the nephroids incribed in a triangle and, among them, those tangent to another given line. This is a sequel to our previous note "Inscribed Cardioids and Eckart Cubics", see [3].

1 Nephroids

The nephroid is a famous tricircular sextic of class 4 having two real finite cusps. It immediately follows that there is generally no nephroid tangent to five given lines. See [1, 2, 7] for further properties.

Following Morley [8], Guy [4] and others, we will focus on nephroids inscribed in the reference triangle ABC and also, among them, those tangent to another given line. We work with barycentric coordinates which is certainly not the most common approach for this type of problem but these are needed when we focus on some related curves.

Note that any center of ABC will be exclusively denoted by its ETC name to avoid any confusion : for example the circumcenter of ABC is X_3 and not (necessarily) O.

Let us denote by :

- $O = p : q : r$, the center of the nephroid $\mathcal{N}$,
- $K_1 = u : v : w$ one of its cusps, K_2 the other cusp,
- $\mathcal{A}$, its axis, the line passing through these three points,
- $\mathcal{M}$, its other axis, the perpendicular bisector of K_1K_2,
- S_1, S_2 its vertices on $\mathcal{M}$,
- $\mathcal{D}_1$, $\mathcal{D}_2$ the perpendiculars at K_1, K_2 to $\mathcal{A}$.

In the sequel, these points O, K_1 are supposed to be finite and distinct.

A simple generation of $\mathcal{N}$ through envelopes of lines is the following. Let S be the reflection of O about K_1 and let M be a variable point on the circle $C(O, S)$ such that $\angle(OS, OM) = t$. The line through M parallel to $\mathcal{A}$ meets this circle again at E and let F be the reflection of E about the line OM. The line $L_M = MF$ envelopes the nephroid. Note that the point of tangency N is the foot of the perpendicular to L_M at the midpoint of OM. See figure 1.

It is then easy to give a parametric tangential equation of $\mathcal{N}$ when the center O and one cusp K_1 are given. With $T = \tan(t/2)$ to avoid trigonometry, an equation of the line L_M can be expressed under the form :

$$(u+v+w)(p+q+r)\Delta(T^2-2T-1)(T^2+2T-1)AX - T[(p+q+r)DR_1 - T^2DR_2], \quad (1)$$

where :

- Δ is the area of ABC,
- $AX = 0$ is an equation of $\mathcal{A}$,

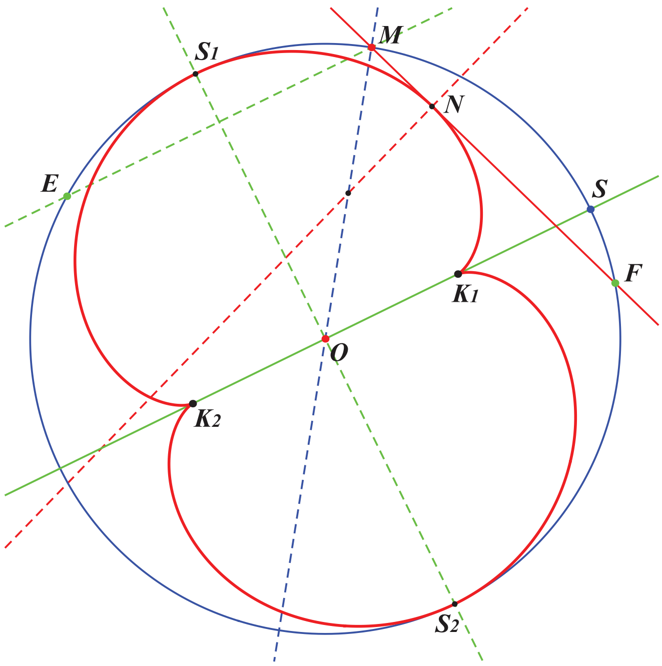

Figure 1: Construction of a nephroid

- $DR_1 = 0$ and $DR_2 = 0$ are the equations of $\mathcal{D}_1$ and $\mathcal{D}_2$.

Remarks :

1. the equation above is clearly of degree 4 in T since the class of $\mathcal{N}$ is 4.
2. when $T = 0$ and $T = \infty$, L_M is the axis $\mathcal{A}$.
3. when $T = \pm 1 \pm \sqrt{2}$, the first term of the equation vanishes and the second term gives the two bitangents.
4. when $T = \pm 1$, M is one of the vertices S_1, S_2 and the equation gives the two tangents at these points.

2 Nephroids tangent to one given line

Let $\mathcal{L}$ be a line, different of the line at infinity, with equation $\alpha\, x + \beta\, y + \gamma\, z = 0$.

When we express that $\mathcal{L}$ is a tangent to the nephroid $\mathcal{N}$ with center O and cusp K_1, we obtain a condition of the form :

$$(p + q + r)\ \Phi_1\ \Phi_2\ \Phi_3^2 - 4\Delta^2\ \Phi_4^2 = 0, \tag{2}$$

where :

- $\Phi_1 = 0$ when $\mathcal{L}$ contains K_1,
- $\Phi_2 = 0$ when $\mathcal{L}$ contains K_2,
- $\Phi_3 = 0$ when $\mathcal{L}$ is perpendicular to the line OK_1,
- $\Phi_4 = 0$ when K_1 lies on a parallel to $\mathcal{L}$ passing through one of the cusps of the nephroid with center O bitangent to $\mathcal{L}$.

If Z is the projection of O on $\mathcal{L}$, then these cusps are the images of Z under the homotheties $h(O, \pm\sqrt{2}/2)$.

This condition (2) is of global degree 4 with respect to all the triads of variables namely $p : q : r$, $u : v : w$ and $\alpha : \beta : \gamma$. If follows that, for a given line $\mathcal{L}$ and a given center O (resp. cusp K_1), the locus of K_1 (resp. O) is a quartic denoted $\mathcal{Q}_O$ (resp. $\mathcal{Q}_K$) represented in figures 2 and 3.

Note that $\mathcal{Q}_O$ must be a central quartic with center O having two parallel asymptotes ($\mathcal{L}$ and its reflection about O) and two nodes at the cusps of the bitangent nephroid as above.

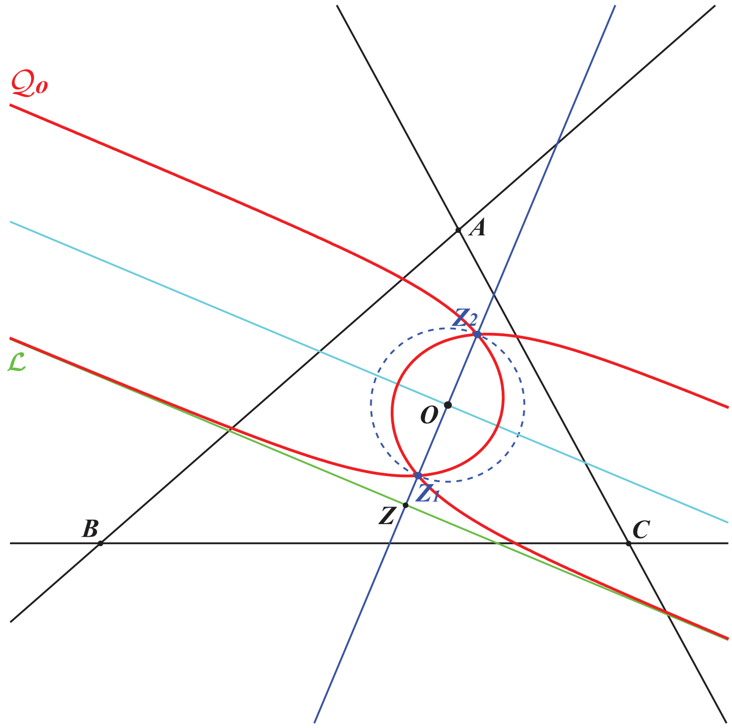

Figure 2: $\mathcal{Q}_O$: locus of cusps of nephroids with center O tangent to $\mathcal{L}$

Construction of $\mathcal{Q}_O$

The center O and the line $\mathcal{L}$ being given, construct successively :

1. the perpendicular at O to $\mathcal{L}$ intersecting $\mathcal{L}$ at Z,
2. a (variable) point k_1 on $\mathcal{C}(O, Z)$,
3. the parallel ℓ at k_1 to $\mathcal{L}$,
4. the perpendicular bisector δ of Ok_1,
5. the intersection K_1 of ℓ and δ,
6. the reflection K_2 of K_1 about O.

Note that the circle with center O passing through the center of the square having OZ as one side intersects OZ at two points Z_1, Z_2 which are the cusps of the nephroid with center O, bitangent to $\mathcal{L}$.

When k_1 varies on $\mathcal{C}(O, Z)$, these points K_1, K_2 lie on $\mathcal{Q}_O$ and then, the nephroid with center O and cusps K_1, K_2 is tangent to $\mathcal{L}$ at a point P which can be constructed as follows.

The axis OK_1K_2 meets $\mathcal{L}$ at T. The parallel at T to Ok_1 meets the perpendicular at O to Ok_1 at a point X. The perpendicular at X to $\mathcal{L}$ meets $\mathcal{L}$ at the requested point P.

From all this, we see that there are generally two nephroids with given center O and given axis passing through O (unless the axis is perpendicular or parallel to $\mathcal{L}$), one being the reflection of the other in the perpendicular at O to $\mathcal{L}$. When the axis is perpendicular to $\mathcal{L}$, we obtain the nephroid bitangent to $\mathcal{L}$ and when it is parallel to $\mathcal{L}$, we obtain that with vertices Z and its reflection Z' about O.

Now, if O, $\mathcal{L}$ and P on $\mathcal{L}$ are given, there is one and only one nephroid with center O tangent at P to $\mathcal{L}$.

Construction of $\mathcal{Q}_K$

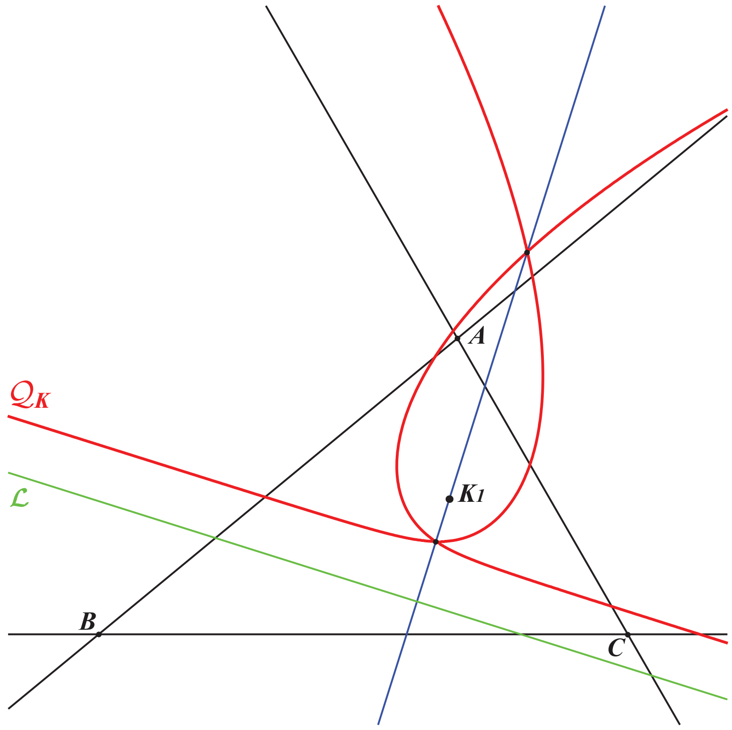

Figure 3: $\mathcal{Q}_K$: locus of centers of nephroids with cusp K_1 tangent to $\mathcal{L}$

A cusp K and the line $\mathcal{L}$ being given, construct successively :

1. a variable line ℓ passing through K and meeting $\mathcal{L}$ at S,
2. the circle γ with diameter KS and center ω,
3. the perpendiculars δ_1, δ_2 at ω to the bisectors at S of the lines $\mathcal{L}$ and ℓ,
4. the intersections E_1, E_2 (resp. E_3, E_4) of γ with δ_1 (resp. δ_2),
5. the two lines passing through K and E_1, E_2 (resp. E_3, E_4) meeting the corresponding bisector at F_1, F_2 (resp. F_3, F_4),
6. the perpendiculars at F_1, F_2 (resp. F_3, F_4) to the corresponding bisector meeting ℓ at O_1, O_2 (resp. O_3, O_4).

These four latter points are the requested centers and lie on $\mathcal{Q}_K$. Note that O_1, O_2 (resp. O_3, O_4) are inverse with respect to γ.

From all this, we see that there are generally four nephroids with given cusp K and given axis ℓ passing through K. When ℓ is perpendicular to $\mathcal{L}$, there are only two of them and both are bitangent to $\mathcal{L}$.

3 Nephroids with given center, tangent to two secant lines

Let L_a, L_b be two lines secant at X and O a point not lying on these lines.

Let γ be the circle with center O, tangent to L_b. The parallels at O to the bisectors of L_a, L_b meet γ at four points $a_i, i \in \{1, 2, 3, 4\}$.

The tangent at a_i to γ meets L_a at A_i and the perpendicular bisector δ of OA_i meets L_a at T_i. The line OT_i is the axis of a nephroid with center O, tangent to L_a, L_b.

Now, the circle with center O, tangent to δ, meets OT_i at two points which are the cusps K_1, K_2 of the nephroid and then the construction is easy. In conclusion, we have

Theorem 1 *There are four nephroids with same center O and tangent to two secant lines L_a and L_b not passing through O.*

Naturally, the eight cusps of these four nephroids lie on the two quartics $\mathcal{Q}_O$ related with O and the two given lines. See figure 4.

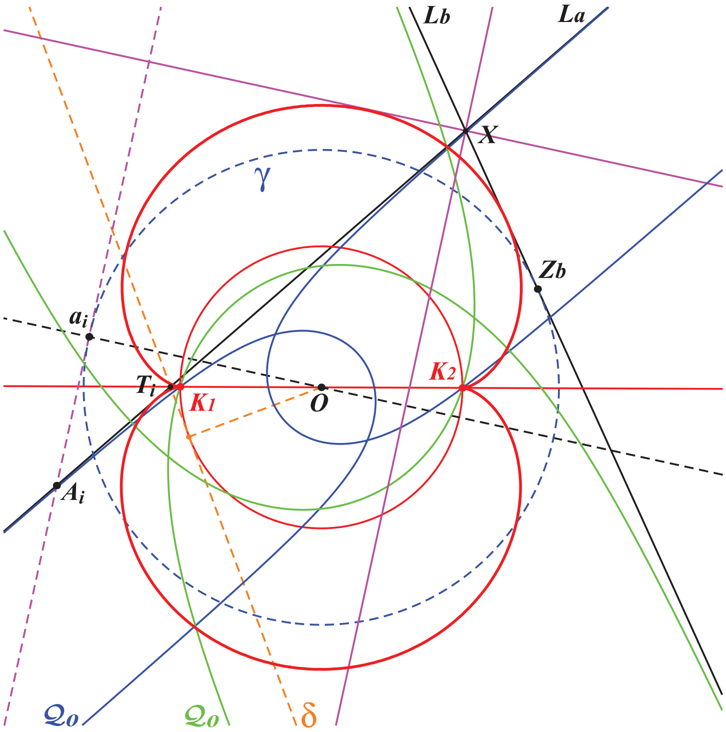

Figure 4: Nephroid with center O, tangent to two lines

Without loss of generality, it is convenient to choose X and the projections of O on L_a, L_b as the vertices of a reference triangle ABC. In this case, O is clearly the antipode of A on the circumcircle of ABC.

After some computations, we obtain

Theorem 2 *The eight cusps of the four nephroids with same center O and tangent to the sidelines $L_a = AB$, $L_b = AC$ lie on a same hyperbola $\mathcal{H}_A$.*

$\mathcal{H}_A$ contains A and its center is O. It also contains the reflection A' of A about O, the intersection Y_a (resp. Y_b) of L_a (resp. L_b) with the reflection of L_b (resp. L_a) about O.

Moreover, the tangent at A contains X_{1092}, the trilinear cube of X_3 which yields its construction.

The asymptotes of $\mathcal{H}_A$ are related to the point X_{255}. One is the parallel at O to the line AX_{255} and the other is the parallel at O to the line passing through A and the trace of the trilinear polar of X_{255} on BC. See figure 5 where one only of the four axes passing through the corresponding cusps K_1, K_2 is represented.

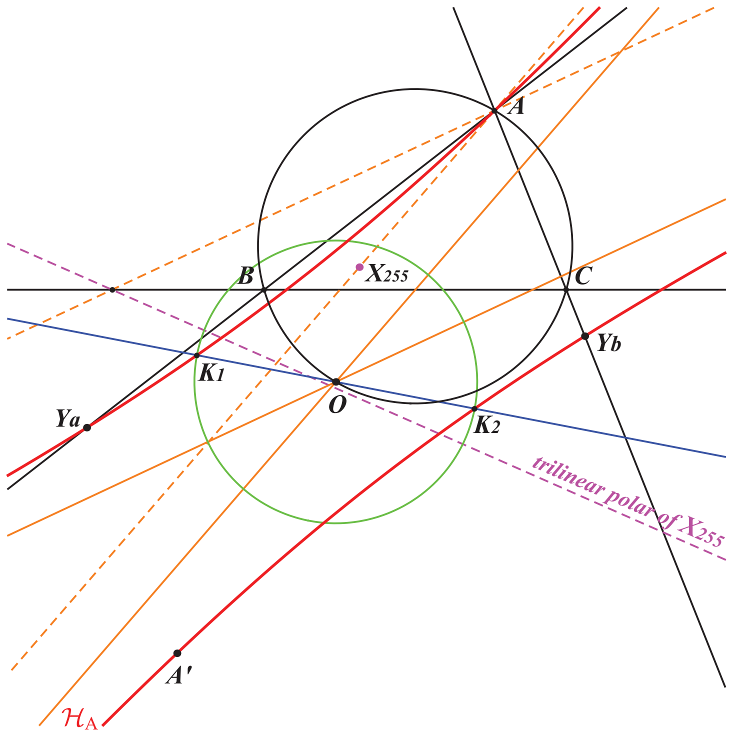

Figure 5: The hyperbola $\mathcal{H}_A$

Now, if L_a, L_b are the two sidelines AB, AC (hence secant at A) and if $O = p : q : r$ is a point not lying on these lines then the equation of the hyperbola $\mathcal{H}_A$ containing the cusps of the four nephroids is

$$b^2 r^4\, y\,[2q(x+z) - (p-q+r)y] - c^2 q^4\, z\,[2r(x+y) - (p+q-r)z] = 0. \tag{3}$$

Note that the tangent at A is the line $b^2 r^3\, y - c^2 q^3\, z = 0$ passing through the point

$$K = \frac{p^3}{a^2} : \frac{q^3}{b^2} : \frac{r^3}{c^2}$$

which is the cusp of the cardioid with center O inscribed in ABC. See [3], §2.

Obviously, if the hyperbolas $\mathcal{H}_B$, $\mathcal{H}_C$ are defined similarly, their respective tangents at B, C also contain K. These three hyperbolas belong to a same pencil of concentric hyperbolas having two pairs of common points, each with midpoint O, and one pair is always real. See figure 6.

4 Nephroids inscribed in ABC

We apply the condition (2) to the sidelines of ABC and find three conditions E_a, E_b, E_c on O and K_1 such that the nephroid $\mathcal{N}$ is inscribed in the reference triangle ABC.

For a given center $O = p : q : r$, E_a gives the central quartic which is locus of the cusps of the nephroids $\mathcal{N}$ tangent to the sideline BC. This has equation :

$$(p+q+r)\; x\; \Psi_1\; \Psi_2^2 - 4\Delta\; \Psi_3^2 = 0, \tag{4}$$

where

- $\Psi_1 = 0$ is the second asymptote i.e. the reflection of BC about O,

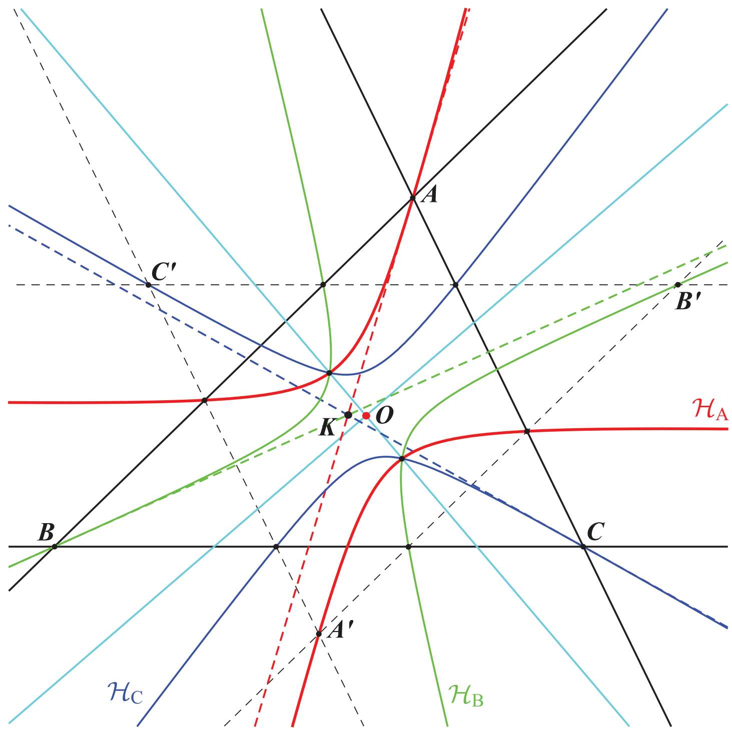

Figure 6: The hyperbolas $\mathcal{H}_A$, $\mathcal{H}_B$, $\mathcal{H}_C$

- $\Psi_2 = 0$ is the perpendicular at O to BC,
- $\Psi_3 = 0$ is the union of the two parallels to BC passing through the cusps Z'_a, Z''_a of the nephroid with center O bitangent to BC.

If Z_a is the projection of O on BC, then these cusps Z'_a, Z''_a are the images of Z_a under the homotheties $h(O, \pm\sqrt{2}/2)$ as mentioned above. See figure 7.

It is then clear that one can find a nephroid inscribed in ABC if and only if the three conditions E_a, E_b, E_c are simultaneously fulfilled which is generally not true for an arbitrary choice of O.

Because of its symmetry, each of the quartics E_a, E_b, E_c can be parametrized using a point $M(t,T) = O + T(1+t : t : -1-2t)$ where t and T are two real numbers. By inserting M into the three equations, we get three polynomials of degree 4 in t and T but of degree 2 in T^2.

The elimination of T^2 between two pairs of these polynomials gives two other conditions of degree 4 in t but each is actually the product of two quadratic polynomials. It is then sufficient to eliminate t between one polynomial of the first condition and one polynomial of the second condition and, after discarding the unsymmetric factors, we find that a nephroid with center O is inscribed in ABC if and only if O lies on one of the four quartics $\mathcal{Q}_o$, $\mathcal{Q}_a$, $\mathcal{Q}_b$, $\mathcal{Q}_c$.

The equation of $\mathcal{Q}_o$ is :

$$\sum_{\text{cyclic}} b^2c^2(-a+b+c)^2\, x^4 - 2a^2bc(a-b+c)(a+b-c)\, y^2z^2 = 0, \tag{5}$$

and those of $\mathcal{Q}_a$, $\mathcal{Q}_b$, $\mathcal{Q}_c$ are obtained under extraversion, replacing successively a, b, c with their opposites.

Each quartic is actually the union of four lines : $\mathcal{Q}_o$ is composed of the trilinear polar of the Yff center of congruence X_{174} and the sidelines of the cevian triangle of this same point. It follows that $\mathcal{Q}_o$ meets each sideline of ABC at two double points. See figure 8.

The corresponding three other quartics give three sets of four lines (obviously obtained under extraversion) and finally we have

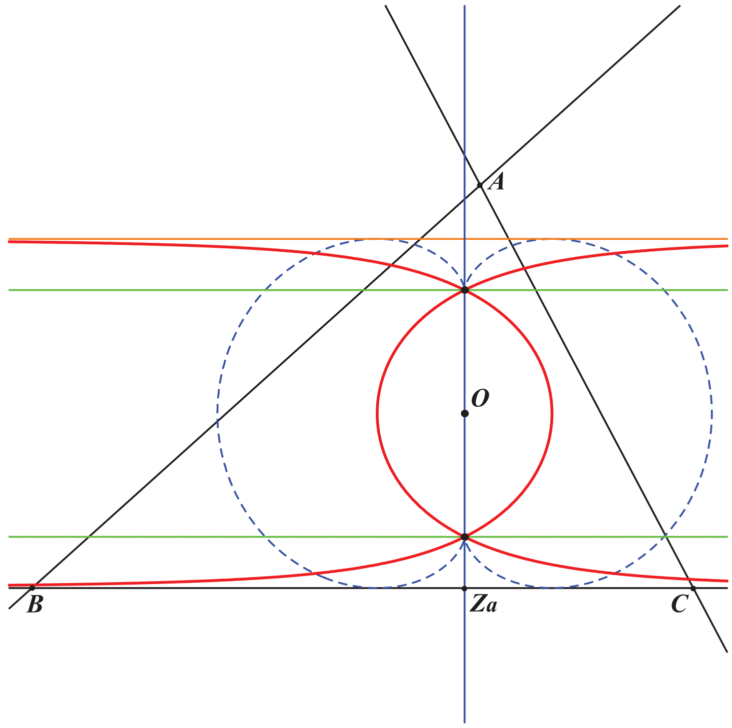

Figure 7: E_a : locus of cusps of nephroids with center O, tangent to BC

Theorem 3 *One can find a nephroid with center O and inscribed in ABC if and only if O lies on one of the 16 lines deduced from X_{174} and its extraversions.*

The existence of these 16 lines is mentioned in [4], page 138, without any further description.

Their construction is fairly easy. The bisectors at the incenter X_1 in the triangle BCX_1 meet BC at two points which are the double points mentioned above. With the double points on CA, AB it is easy to draw the requested four lines of the first set, each passing through three double points. To obtain the three remaining sets, it is enough to replace X_1 with an excenter.

$\mathcal{Q}_o$ can also be seen as the locus of the square roots of points that lie on the inscribed ellipse $\mathcal{E}_o$ with perspector X_{57} passing through X_{2170}, X_{3020}. Hence, if P is any point on the line at infinity, the barycentric product $X_{174} \times P$ is a point on $\mathcal{Q}_o$ and so are its harmonic associates (the vertices of the anticevian triangle of $X_{174} \times P$).

Thus, any point O on one of the 16 lines is associated with three quartics E_a, E_b, E_c intersecting at two points K_1, K_2 which are the cusps of the nephroid with center O and inscribed in ABC. The figure 8 above shows these curves and points when O is a point on the trilinear polar of X_{174}.

Construction of the inscribed nephroid with given center O

The triangle ABC and the center O (lying on one of the 16 lines above) being given, it is sufficient to construct the two sets of four axes of the nephroids tangent to two sidelines of ABC as in §3 and choose the axis common to the two sets.

5 Nephroids inscribed in ABC and tangent to a fourth line

Let L be a line which is not the line at infinity nor one of the sidelines of the reference triangle ABC.

Any nephroid inscribed in ABC and tangent to L must have its center O on one of the 16 lines above and on the quartic $\mathcal{Q}_O$ related with L as in §2. It follows that there are generally 64 such nephroids since each line meets $\mathcal{Q}_O$ at four points.

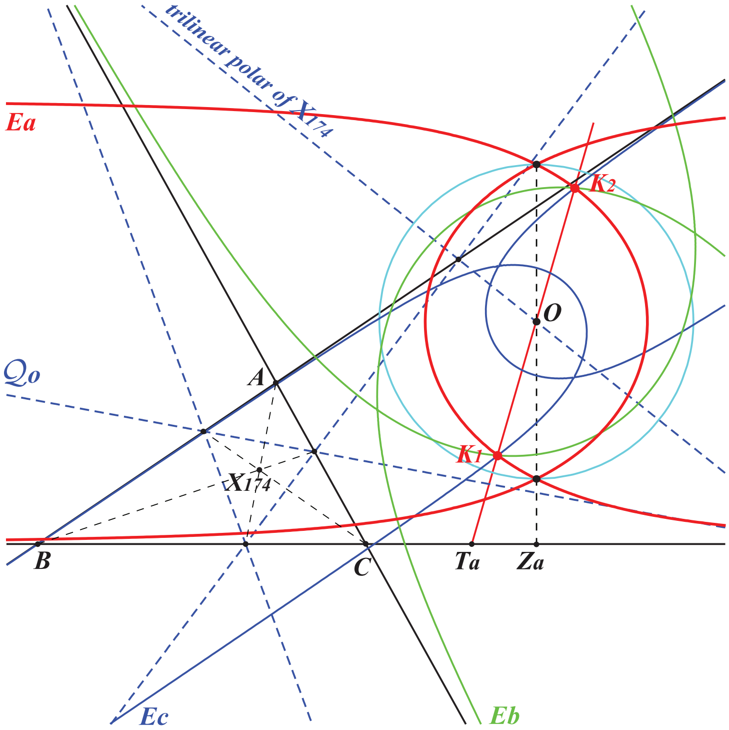

Figure 8: $\mathcal{Q}_o$ with E_a, E_b, E_c

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