Equivalent Pivotal Cubics

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1 Anharmonic ratios on pivotal cubics

Denote by $p\mathcal{K} = p\mathcal{K}(\Omega, P)$ the pivotal cubic with pole $\Omega = p : q : r$ and pivot $P = u : v : w$, the locus of point M such that P, M and its Ω-isoconjugate M^* are colinear.

It is a known fact that the tangents at A, B, C, P to the cubic concur at P^*. These tangents meet any line (not passing through P^*) at four points M_a, M_b, M_c, M_p and the anharmonic ratio (or cross-ratio) of these tangents does not depend on the line. It is given by

$$\lambda = \frac{q(ru^2 - pw^2)}{p(rv^2 - qw^2)} = \frac{\dfrac{u^2}{p} - \dfrac{w^2}{r}}{\dfrac{v^2}{q} - \dfrac{w^2}{r}},$$

assuming that the four tangents are taken in the order above.

If we change this order, this anharmonic ratio will always be in the set

$$\left\{\lambda, \frac{1}{\lambda}, \lambda - 1, \frac{1}{\lambda - 1}, \frac{\lambda}{\lambda - 1}, \frac{\lambda - 1}{\lambda}\right\}.$$

Furthermore, λ is not altered when P^* is replaced by any point on the cubic and the four tangents by the tangents drawn from this point to the cubic (Salmon). Hence, this anharmonic ratio is an invariant of any pivotal cubic which we call the anharmonic ratio of the cubic and denote by $\lambda(\Omega, P)$.

This anharmonic ratio is unaltered by projection or any other linear transformation such as any homography in the plane fixing A, B, C and mapping the pivot P to another point P_1.

For example, the Thomson cubic $p\mathcal{K}(K, G)$ and the Lucas cubic $p\mathcal{K}(G, H')$ (H' is the isotomic conjugate of H) have the same anharmonic ratio since the latter cubic is the anticomplement of the former one.

Nevertheless, the reciprocal is false since we can find non linear transformations keeping the anharmonic ratio of a $p\mathcal{K}$ unaltered under the transformation.

For example, any isoconjugation transforms a $p\mathcal{K}$ into another $p\mathcal{K}$ having the same anharmonic ratio. Indeed, the isoconjugation with pole $\alpha : \beta : \gamma$ maps $p\mathcal{K}(\Omega, P)$ to the $p\mathcal{K}$ with pole $\dfrac{\alpha^2}{p} : \dfrac{\beta^2}{q} : \dfrac{\gamma^2}{r}$ and pivot $\dfrac{u\alpha}{p} : \dfrac{v\beta}{q} : \dfrac{w\gamma}{r}$ which clearly shows the result.

2 Equivalent pivotal cubics

Let us now consider two pivotal cubics $p\mathcal{K} = p\mathcal{K}(\Omega, P)$ and $p\mathcal{K}_1 = p\mathcal{K}(\Omega_1, P_1)$. We shall say that these cubics are (anharmonically) equivalent if and only if $\lambda(\Omega, P) = \lambda(\Omega_1, P_1)$ which gives the condition

$$\sum_{\text{cyclic}} \frac{u_1^2}{p_1} p\,(rv^2 - qw^2) = 0 \iff \sum_{\text{cyclic}} \frac{p}{p_1} \left[\left(\frac{q}{v^2} - \frac{r}{w^2} \right) \left(\frac{u_1}{u} \right)^2 \right] = 0 \qquad (1)$$

where $\Omega_1 = p_1 : q_1 : r_1$ and $P_1 = u_1 : v_1 : w_1$.

2.1 Special cases

This condition simplifies when the two cubics have the same pole or the same pivot and this gives the two following corollaries.

Corollary 1 *Two pivotal isocubics with the same pole Ω are equivalent if and only if the pivot P_1 of one of them lies on the diagonal conic passing through the fixed points of the isoconjugation and the pivot P of the other one.*

This conic is actually the polar conic of P in $p\mathcal{K}(\Omega, P)$. It contains the vertices of the anticevian triangle of P, the cevian quotient P^*/P and the vertices of the anticevian triangle of P^*/P. Its center is the tripole of the line passing through Ω and the barycentric square P^2 of P.[1]

The two equivalent $p\mathcal{K}$s have seven known common points (A, B, C, the four (real or not) fixed points of the isoconjugation with pole Ω), hence they have two other common points which are the (real or not) intersections of the line PP_1 and the circum-conic passing through P^*, P_1^*. These points Z, Z^* are isoconjugate points on both cubics.

Construction 1 : equivalent $p\mathcal{K}$s with the same pole Ω

[1] If $\Omega = P^2$, the cubic decomposes into the cevian lines of P.

We take two pivots P, P_1 on the same diagonal conic as described above and another point E on this conic. The line EP (resp. EP_1) meets the circum-conic through E^* and P^* (resp. P_1^*) at two isoconjugate points M, M^* (resp. N, N^*) such that the locus of M (resp. N) is $p\mathcal{K}(\Omega, P)$ (resp. $p\mathcal{K}(\Omega, P_1)$).

From this, we see that we can find two mappings from $p\mathcal{K}(\Omega, P)$ onto $p\mathcal{K}(\Omega, P_1)$: one maps M onto N (hence M^* onto N^*) and the other M onto N^* (hence M^* onto N) which give the following colinearities :

$$Z, M, N \ ; \ Z, M^*, N^* \ ; \ Z^*, M, N^* \ ; \ Z^*, M^*, N.$$

Example 1 : Any isogonal $p\mathcal{K}$ equivalent to the Thomson cubic has its pivot on the Steiner rectangular hyperbola i.e. the diagonal conic passing through the in/excenters and G, with center the Steiner point X_{99}. Since the conic contains X_{20}, the Darboux cubic is equivalent to the Thomson cubic.[2] In this case, Z, Z^* are real : they are O and H.

If we take the infinite points of the Steiner rectangular hyperbola as pivots, we find two circular isogonal $p\mathcal{K}$s equivalent to the Thomson cubic and the points Z, Z^* are the foci (two real, two imaginary) of the Steiner in-ellipse.

Corollary 2 *Two pivotal isocubics with the same pivot P are equivalent if and only if the pole of one of them lies on the circum-conic passing through the pole of the other one and the barycentric square of the common pivot.*

Construction 2 : equivalent $p\mathcal{K}$s with the same pivot P

We take two poles Ω_1, Ω_2 on the same circum-conic through P^2 and we denote by P_1, P_2 the isoconjugates of P in these isoconjugations. For any Y on the line P_1P_2, let Y_1, Y_2 be its isoconjugates in the same isoconjugations. The circum-conic passing through Y and P_1 (resp. P_2) meets the line Y_1Y_2 at two points on $p\mathcal{K}(\Omega_1, P)$ (resp. $p\mathcal{K}(\Omega_2, P)$).

These two cubics contain seven known points (A, B, C, P, the traces P_a, P_b, P_c of P) and two (not necessarily real) other points Z_1, Z_2 which are the common points of the line P_1P_2 and the conic through P_a, P_b, P_c, P/P_1, P/P_2. These points Z_1, Z_2 are homologous in the isoconjugation with fixed point P with respect to the cevian triangle of P and, more over, each one is the cevian quotient of P and the other one. At last, the two cubics are two

[2]The conic also contains X_{63}, X_{147}, X_{194}, X_{487}, X_{488}, X_{616}, X_{617}, X_{627}, X_{628}, X_{1670}, X_{1671}, X_{1764}, X_{2128}, X_{2582}, X_{2583}, X_{2896}. This gives a selection of isogonal $p\mathcal{K}s$ equivalent to the Thomson cubic.

$p\mathcal{K}$s with pivots P_1, P_2 with respect to the triangle $P_aP_bP_c$, invariant in the isoconjugation cited above.

Example 2 : Any $p\mathcal{K}$ with pivot G equivalent to the Thomson cubic has its pole on the circum-conic through G and K.[3]

For example, $p\mathcal{K}(X_{37}, G)$ is equivalent to the Thomson cubic. The points Z_1, Z_2 are the incenter $I = X_1$ and the Mittenpunkt X_9.[4] $p\mathcal{K}(X_{37}, G)$ is the complement of **K034** $= p\mathcal{K}(G, X_{75})$ (Spieker perspector cubic), also equivalent to the Thomson cubic (see example 3 below).

2.2 Back to the general case

The equation (1) shows that, for a given pivot P_1, the pole Ω_1 must lie on the circum-conic Γ_1 with perspector $W_1 = u_1^2\, p\,(rv^2 - qw^2) ::$, this conic containing P_1^2 and the point $Q_1 = \dfrac{u_1}{v_1 - w_1}\, p\,(rv^2 - qw^2) ::$.

Note that, for any Ω_1 on the conic, the points Ω_1, Q_1, P_1^* are colinear and the locus of P_1^* is the circum-conic Γ_2 with perspector $W_2 = u_1\, p\,(rv^2 - qw^2) ::$ passing through P_1 and Q_1. From this, Q_1 is the trilinear pole of the line W_1W_2.

Let $Z = u\, p\,(rv^2 - qw^2) ::$ be the intersection of the trilinear polars of P and its Ω-isoconjugate P^* i.e. the perspector of the circum-conic passing through P and P^*. Denote by f the homography which fixes A, B, C and maps the pivot P to this point Z.

f maps P_1^2 and P_1 to the points W_1 and W_2.

Construction of $M' = f(M)$

Let $P_aP_bP_c$, $Z_aZ_bZ_c$, $M_aM_bM_c$ be the cevian triangles of P, Z, M.

The parallel at B to AP meets AM at E, the parallel at Z_a to AP meets AC at E'. EE' meets BC at $M'_a = f(M_a)$ hence M' lies on the line AM'_a. With two similar constructions, we see that M' is the common point of the lines AM'_a, BM'_b, CM'_c.

Any point Ω_1 on Γ_1 can be seen as the W_1-isoconjugate of an infinite point $\alpha : \beta : \gamma$, hence we can write

$$\Omega_1 = \frac{u_1^2\, p\,(rv^2 - qw^2)}{\alpha} ::$$

[3]The conic also contains X_{25}, X_{37}, X_{42}, X_{111}, X_{251}, X_{263}, X_{308}, X_{393}, X_{493}, X_{494}, X_{588}, X_{589}, X_{694}, X_{941}, X_{967}, X_{1169}, X_{1171}, X_{1218}, X_{1239}, X_{1241}, X_{1383}, X_{1400}, X_{1427}, X_{1880}, X_{1976}, X_{1989}, X_{2054}, X_{2165}, X_{2248}, X_{2350}, X_{2395}, X_{2433}, X_{2963}, X_{2981}, X_{2987}, X_{2998}. This gives another selection of $p\mathcal{K}s$ with pivot G equivalent to the Thomson cubic.

[4]$p\mathcal{K}(X_{37}, G)$ contains the points I, G, X_9, X_{10}, X_{37}, X_{226}, X_{281}, X_{1214}.

and the equation of any $p\mathcal{K}$ with pivot P_1 equivalent to $p\mathcal{K}(\Omega, P)$ takes the form

$$\sum_{\text{cyclic}} \frac{1}{\alpha}\, p\,(rv^2 - qw^2)u_1^2(w_1 y - v_1 z)yz = 0. \tag{2}$$

The same equation (1) shows that, for a given pole $\Omega_1 = p_1 : q_1 : r_1$, the pivot must lie on the diagonal conic with equation

$$\sum_{\text{cyclic}} \frac{p}{p_1}\,(rv^2 - qw^2)x^2 = 0, \tag{3}$$

passing through the fixed points of the isoconjugation with pole Ω_1 and also the fixed points of the isoconjugation with pole the Ω_1-isoconjugate of the Ω-isoconjugate of P^2. Its center is

$$f^{-1}(\Omega_1) = \frac{p_1}{p(rv^2 - qw^2)} :: .$$

Example 3 : Any isotomic $p\mathcal{K}$ equivalent to the Thomson cubic has its pivot on the diagonal conic passing through G, the vertices of the antimedial triangle and X_{69}.[5] With X_{69}, we obtain the Lucas cubic and with X_{75}, we obtain **K034** $= p\mathcal{K}(G, X_{75})$ (Spieker perspector cubic).

Example 4 : Any $p\mathcal{K}$ with pole X_{32} equivalent to the Thomson cubic has its pivot on the Stammler hyperbola passing through the vertices of the excentral and tangential triangles, containing a large number of centers.[6] With X_1, we obtain **K175** $= p\mathcal{K}(X_{32}, X_1)$ and with X_3, we obtain **K172** $= p\mathcal{K}(X_{32}, X_3)$.

[5] This conic also contains X_{75}, X_{1272}, X_{1369}, X_{1370}.

[6] for example X_1, X_3, X_6, X_{155}, X_{159}, X_{195}, X_{399}, X_{610}, X_{1498}, X_{1740}, X_{2574}, X_{2575}, X_{2916}, X_{2917}, X_{2918}, X_{2929}, X_{2930}, X_{2931}, X_{2935}, X_{2948}.