Isometric Parallel Chords on the Neuberg Cubic

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Abstract

We study the properties of chords of same length on the Neuberg cubic which are parallel to the Euler line. We shall meet several interesting related cubics. This paper is a sequel to our previous paper [4].

1 Chords on the Neuberg cubic

1.1 Generalities

Let Q be the point on the Euler line defined by $\overrightarrow{OQ} = k\,\overrightarrow{OH}$ where O and H are the circumcenter and orthocenter of the reference triangle ABC and k a number in $\mathbb{R} \cup \{\infty\}$.

We know that any point P on the Neuberg cubic K001 is characterized by the fact that the line passing through P and its isogonal conjugate P^* is parallel to the Euler line or, equivalently, contains X_{30}.

When we express that $\overrightarrow{PP^*} = \overrightarrow{OQ} = k\,\overrightarrow{OH}$ we find that P must lie on three rectangular hyperbolas $\mathcal{H}_A$, $\mathcal{H}_B$, $\mathcal{H}_C$ passing through A, B, C respectively and belonging to a same pencil. See figure 1. This gives

Proposition 1 *For a given k, there are at most four real points P_i (and four real points $Q_i = P_i^*$) on* K001 *such that the chords P_iQ_i are parallel to the Euler line and have the same length $|k| \times OH$.*

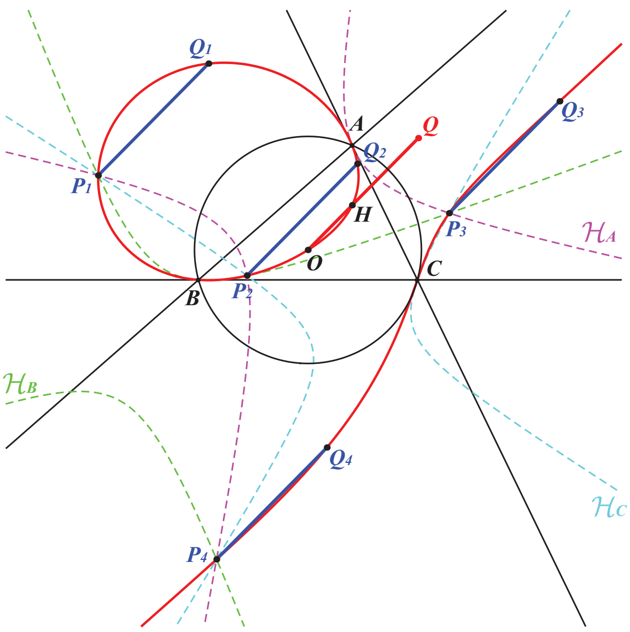

Figure 1: The rectangular hyperbolas $\mathcal{H}_A$, $\mathcal{H}_B$, $\mathcal{H}_C$

Construction of $\mathcal{H}_A$

$\mathcal{H}_A$ has its asymptotes parallel to the bisectors at A. The tangent at A passes through X_{74}. Its center O_a is the midpoint of AS_a where S_a is defined by $\overrightarrow{AS_a} = \overrightarrow{QO}$. See figure 2.

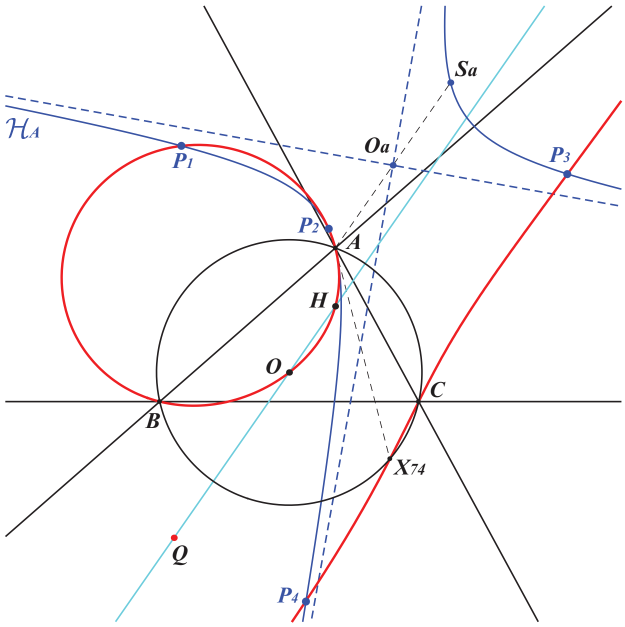

Figure 2: Construction of the rectangular hyperbola $\mathcal{H}_A$

Remarks :

1. when $k = 0$, these chords are each reduced to a single point namely an in/excenter of ABC. This trivial case is excluded in the sequel.

2. two of these chords are always real and have their end points on the inflexional branch of K001.

3. the remaining two chords have their end points on the oval branch and they are real for certain "not too large" values of k. They can even coincide. This is discussed below in §1.2.

4. the midpoint of any chord lies on the polar conic of X_{30} in the Neuberg cubic. This is the rectangular diagonal hyperbola $\mathcal{D}$ passing through X_1, X_5, X_{30}, X_{395}, X_{396}, X_{523}, X_{1749}, the excenters. Its center is X_{476}.

Proposition 2 *For a given k, the points P_i are the common points of two rectangular hyperbolas $\mathcal{H}_J$ and $\mathcal{H}_K$ homothetic to the Jerabek and Kiepert hyperbolas respectively.*

More precisely, let M be the midpoint of OQ, Ω_k the reflection of M about O and τ the translation that maps O onto Ω_k. Any rectangular hyperbola of the pencil above has its center on the image $\tau(\mathcal{O})$ of the circumcircle $\mathcal{O}$ under τ.

$\mathcal{H}_J$ is the rectangular hyperbola with center $\tau(X_{110})$. It contains O, X_{399} (the reflection of O about X_{110}), the reflection Q' of Q about O and its asymptotes are parallel to those of the Jerabek hyperbola.

$\mathcal{H}_K$ is the rectangular hyperbola with center $\tau(X_{99})$. It contains X_{616}, X_{617} (anticomplements of the Fermat points) and its asymptotes are parallel to those of the Kiepert hyperbola.

Note that O, X_{399} and X_{616}, X_{617} lie on K001 and each pair of these points defines a line passing through X_{74}. In other words, for any k, the coresidual of the four points P_i is X_{74}. See figure 3.

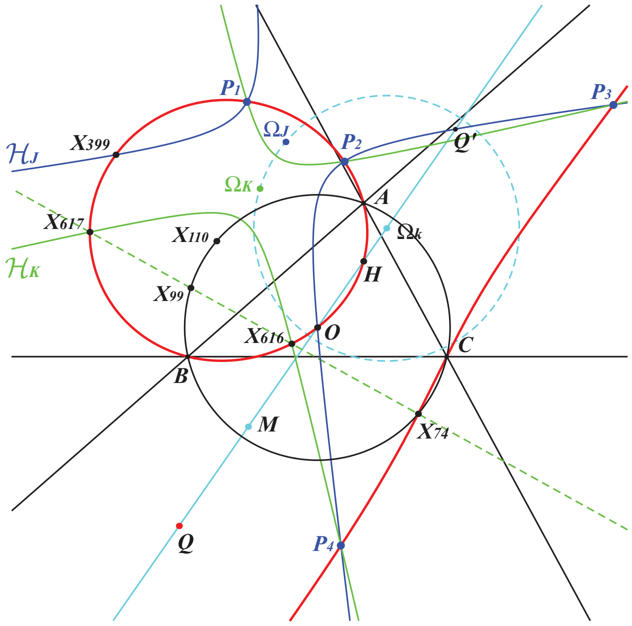

Figure 3: The rectangular hyperbolas $\mathcal{H}_J$ and $\mathcal{H}_K$

Remarks :

1. $\mathcal{H}_J$ decomposes into two perpendicular lines (the parallels at O and X_{399} to the asymptotes of the Jerabek hyperbola) when

$$k = \pm\frac{2R}{OH} = \pm\frac{2}{\sqrt{1 - 8\cos A \cos B \cos C}}.$$

In this case, the corresponding points Q lie on the circle with center O passing through X_{399} .

2. $\mathcal{H}_K$ decomposes into two perpendicular lines (the parallels at X_{616} and X_{617} to the asymptotes of the Kiepert hyperbola) when

$$k = \pm\frac{4}{\sqrt{3(\cot^2\omega - 3)}}, \text{ where } \omega \text{ is the Brocard angle.}$$

1.2 The longest chord on the oval of the Neuberg cubic

Recall that the Neuberg cubic is always composed of two parts : one is an infinite branch containing the real points of inflexion and X_{74} (its intersection with its real asymptote), the other is an oval and the singular focus X_{110} is always inside this oval branch.

Each branch is entirely contained in a region delimited with two parallels to the Euler line passing through two of the in/excenters. See figure 4.

Without loss of generality, we may suppose $a > b > c$ (as in the figure) in which case the oval is contained in the region delimited with the parallels at the incenter I and the excenter I_b. It is then clear by continuity that a chord parallel to the Euler line and contained in this region must reach a maximum length.

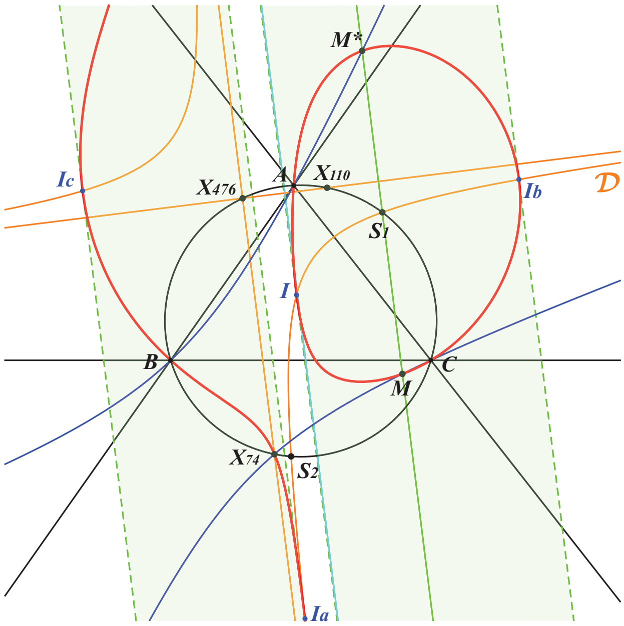

Figure 4: The longest chord

The polar conic $\mathcal{D}$ with center X_{476} passing through the in/excenters and X_5 meets the circumcircle $\mathcal{O}$ at two and only two real points S_1, S_2. Recall that any chord parallel to the Euler line must have its midpoint on $\mathcal{D}$. One of these two real points, say S_2, does not lie in the two regions above and then cannot be the midpoint of a chord.

The other point S_1 is the requested point giving the midpoint of the chord MM^* of maximum length d on the oval of the Neuberg cubic.

The proof is obtained through computation involving the symmetric elementary fonctions of the roots of fourth degree polynomials. Here are some elements of this proof.

Let S be the midpoint of X_{110}, X_{476}, a point on the Euler line which is the orthic line of the Neuberg cubic. Recall that X_{110} is its singular focus and that X_{476} is the center of the polar conic $\mathcal{D}$ of X_{30}. X_{476} lies on the real asymptote of the cubic.

Let P be the point such that $\overrightarrow{SP} = T\ \overrightarrow{SX_{110}}$ where T is a real number. The parallel L at P to the Euler line meets its isogonal transform at two points M, M^* on the Neuberg cubic and let $f(T)$ be the square of the distance MM^*. After some calculations we find that

$$f(T) = K \times \frac{(T - OH^2\, T_o)(T - OH^2\, T_a)(T - OH^2\, T_b)(T - OH^2\, T_c)}{(1+T)^2}$$

where

- $T_o = -\dfrac{a+b+c}{(b+c)(c+a)(a+b)}$,
- T_a, T_b, T_c are the extraversions of T_o,
- K is a constant defined by $K = -\dfrac{(b^2-c^2)^2(c^2-a^2)^2(a^2-b^2)^2}{16\Delta^2\, OH^6} < 0$.

It is easy to verify that $T_o + T_a + T_b + T_c = 0$ and that $T_o \times T_a \times T_b \times T_c < 0$. Obviously, $f(T) = 0$ when L contains an in/excenter.

In order to find the maximum of the distance MM^* we compute the derivative of f and obtain an expression of the form

$$f'(T) = K_1 \times \frac{g(T)}{(1+T)^3}$$

where K_1 is a constant and $g(T)$ a polynomial of degree 4.

Now, if we express that L and $\mathcal{D}$ intersect again on the circumcircle we obtain precisely the condition $g(T) = 0$. In other words, f reaches an extremum if and only if L passes through one of the four intersections of $\mathcal{D}$ and the circumcircle but only one corresponds to a maximum.

2 Circum-cubics $\mathcal{K}(k)$ passing through the points P_i

2.1 Definition

Consider the decomposed cubic $\mathcal{K}_A$ which is the union of the rectangular hyperbola $\mathcal{H}_A$ and the sideline BC. Define $\mathcal{K}_B$, $\mathcal{K}_C$ similarly and let $\mathcal{K}(k)$ be the cubic which is the sum of these three cubics.

Let Δ be the area of ABC. An easy computation shows that the equation of $\mathcal{K}(k)$ takes the form

$$16\Delta^2 \sum_{\text{cyclic}} x(c^2y^2 - b^2z^2) - 3k(x+y+z) \sum_{\text{cyclic}} a^2(b^2-c^2)(-a^2+b^2+c^2)yz = 0,$$

where

$$\sum_{\text{cyclic}} x(c^2y^2 - b^2z^2) = 0 \text{ and } \sum_{\text{cyclic}} a^2(b^2-c^2)(-a^2+b^2+c^2)yz = 0$$

are the equations of the Thomson cubic K002 and the Jerabek hyperbola respectively.

All these cubics $\mathcal{K}(k)$ form a pencil whose nine base-points are A, B, C, O, H, K and the infinite points of the Thomson cubic (which is obtained when $k = 0$).

Furthermore, $\mathcal{K}(k)$ meets the Neuberg cubic at A, B, C, O, H and the four points P_i hence, for a given k, these two cubics generate a pencil $\mathcal{F}(k)$ with nine identified base-points. This pencil is studied in §3.

$\mathcal{K}(k)$ meets the Euler line and Brocard axis again at two points E_3, B_3 defined by

$$\overrightarrow{OE_3} = \frac{1}{3}(1-3k)\,\overrightarrow{OH} \text{ and } \overrightarrow{OB_3} = \frac{1}{2}(2-3k)\,\overrightarrow{OH},$$

and the line E_3B_3 envelopes the parabola $\mathcal{P}$ with focus X_{691} (on the circumcircle), directrix the line $X_{99}X_{110}$. $\mathcal{P}$ is inscribed in the triangle GOK. The contacts with the sidelines GO, OK, GK are X_{376}, X_{182}, X_{1992} respectively. See figure 5.

$\mathcal{K}(k)$ meets the circumcircle at the same points as the isogonal pivotal cubic whose pivot is actually the point E_3 above. For example, $\mathcal{K}(1/3)$, $\mathcal{K}(-1/6)$, $\mathcal{K}(-2/3)$ meet the circumcircle at the same points as K003, K005, K006 respectively but the most remarkable is probably $\mathcal{K}(4/3) =$ K047, a central cubic with center O for which $Q = X_{20}$. See §2.2 for further details and a figure.

2.2 Special cases

Recall that $\mathcal{K}(0)$ is the Thomson cubic in which case the points P_i are the in/excenters of ABC. This is the only pivotal cubic $\mathcal{K}(k)$ and it can be easily seen that $\mathcal{K}(k)$ cannot be a $ps\mathcal{K}$, a circular or equilateral cubic.

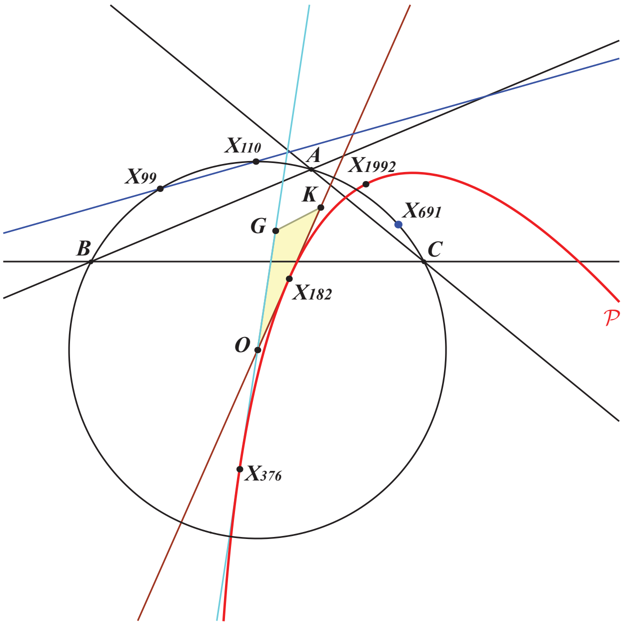

Figure 5: The parabola $\mathcal{P}$

There is one and only one cubic $\mathcal{K}(k)$ with concurring asymptotes and it is obtained when $k = 4/3$. This cubic is the central cubic $\mathcal{K}(4/3) =$ K047 with center O meeting the Neuberg cubic at four points P_i. Its isogonal transform is K615 meeting the Neuberg cubic at four points Q_i, isogonal conjugates of the points P_i, such that $P_iQ_i = 4\,OG$. See figure 6.

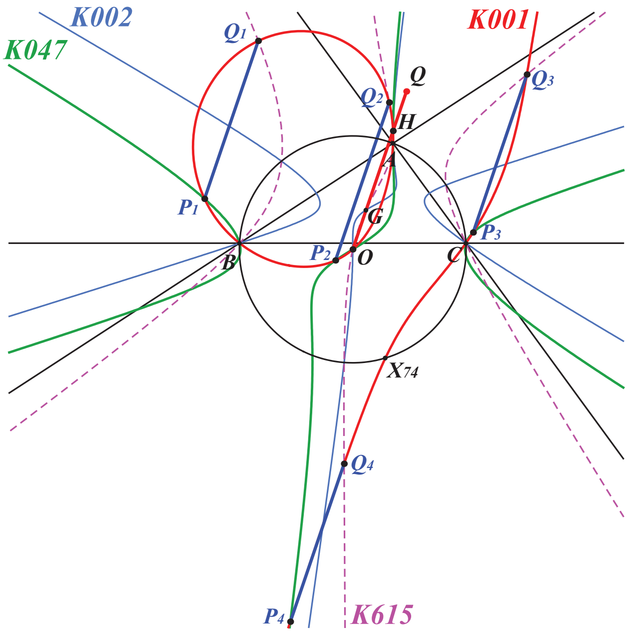

Figure 6: K001 and the central cubic K047

There are two cubics $\mathcal{K}(k)$ which are non-pivotal cubics namely $n\mathcal{K}_1$ and $n\mathcal{K}_2$. They are obtained when Q is an intersection X_{2479}, X_{2480} of the Euler line and the Steiner ellipse. Their isogonal transforms are $n\mathcal{K}_1^*$ and $n\mathcal{K}_2^*$, which are also non-pivotal cubics. See figure 7.

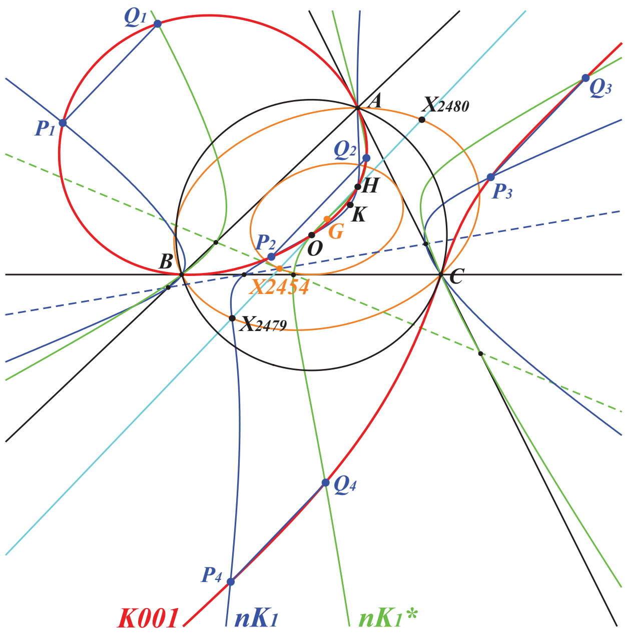

Figure 7: K001 and the non-pivotal cubic $n\mathcal{K}_1$

3 The pencil $\mathcal{F}(k)$

Recall that $\mathcal{F}(k)$ is the pencil of cubics generated by the Neuberg cubic and the cubic $\mathcal{K}(k)$.

In particular, $\mathcal{F}(0)$ is the Euler pencil and all the cubics of $\mathcal{F}(0)$ are isogonal pivotal cubics. This is the only value of k for which $\mathcal{F}(k)$ contains a pivotal cubic.

Now, for $k \neq 0$, any pencil $\mathcal{F}(k)$ contains four (not necessarily all distinct) special cubics namely

1. one stelloid $\mathcal{S}(k)$,
2. one central cubic $\mathcal{C}(k)$,
3. two pseudo-pivotal cubics $\mathcal{P}_1(k)$, $\mathcal{P}_2(k)$.

These cubics are studied in the next paragraphs. See also the table at the end of this section.

3.1 The stelloid $\mathcal{S}(k)$

For any $k \neq 0$, $\mathcal{S}(k)$ is a member of the pencil of cubics generated by the McCay cubic K003 and the union of the line at infinity $\mathcal{L}_\infty$ and the Jerabek hyperbola $\mathcal{J}$. The nine base-points are A, B, C, O, H (counted twice since the tangent contains X_{51}) and the infinite points of K003.

It follows that $\mathcal{S}(k)$ has three real asymptotes parallel to those of the McCay cubic and concurring at the point X on the Euler line $\mathcal{E}$ defined by $\overrightarrow{OX} = (1-k)/3\,\overrightarrow{OH} = k\,\overrightarrow{GO}$. This point X lies on the stelloid when $k = -2, -1/2, 1$ giving the three central stelloids K525, K026, K080 respectively.

$\mathcal{S}(k)$ meets the circumcircle at the same points as $p\mathcal{K}(X_6, Y)$ where Y is the reflection of Q about O. This point Y lies on $\mathcal{S}(k)$.

3.2 The central cubic $\mathcal{C}(k)$

For any $k \neq 0$, $\mathcal{C}(k)$ is a member of the pencil of cubics generated by the Darboux cubic K004 and the union of the Euler line and the circumcircle.

Hence, the center is O and the cubic always contains the de Longchamps point X_{20}. The remaining base-points are obviously H, A, B, C and their reflections A', B', C' about O.

The asymptotic directions of $\mathcal{C}(k)$ are those of $p\mathcal{K}(X_6, S)$ where S is the reflection of H about the midpoint of OQ also defined by $\overrightarrow{OS} = (k-1)\,\overrightarrow{OH}$.

3.3 The two pseudo-pivotal cubics $\mathcal{P}_1(k)$, $\mathcal{P}_2(k)$

See [3] for definition and properties of pseudo-pivotal cubics $ps\mathcal{K}$.

$\mathcal{P}_1(k) = ps\mathcal{K}(\Omega_1, P_1, X_3)$ is a member of the pencil generated by the Thomson cubic K002 and the third Musselman cubic K028 $= ps\mathcal{K}(X_4, X_{264}, X_3)$, a stelloid. The base-points are A, B, C, H (each twice) and O. The tangents at A, B, C are the symmedians and the tangent at H passes through X_{64}.

$\mathcal{P}_2(k) = ps\mathcal{K}(\Omega_2, X_2, X_3)$ is a member of the pencil generated by the Thomson cubic K002 and the first Musselman cubic K026 $= ps\mathcal{K}(X_{51}, X_2, X_3)$, a central cubic. The base-points are A, B, C, the midpoints of ABC, O (twice), H. The tangent at O passes through X_{64}.

The following table gives some characteristics of these two cubics.

	$\mathcal{P}_1(k)$	$\mathcal{P}_2(k)$
third point on OH	$\overrightarrow{OY_1} = (1-2k)/3\,\overrightarrow{OH}$	$\overrightarrow{OY_2} = (1-k)/3\,\overrightarrow{OH}$
associated pivots	$\overrightarrow{OS_1} = (1+k)/3\,\overrightarrow{OH}$	$\overrightarrow{OS_2} = (1+2k)/3\,\overrightarrow{OH}$
points on $\mathcal{L}_\infty$	those of $p\mathcal{K}(X_6, S_1)$	those of $p\mathcal{K}(X_6, S_2)$
points on $\mathcal{O}$	those of $p\mathcal{K}(X_6, Y_1)$	those of $p\mathcal{K}(X_6, Y_2)$
pseudo-pole	Ω_1 = isogonal of S_1	$\Omega_2 = X_6 \times Y_2$
pseudo-pivot	P_1 = isotomic of Ω_1	G
pseudo-isopivot	X_6	Ω_2

Remarks :

1. Ω_1 lies on the Jerabek hyperbola and on $\mathcal{P}_1(k)$.
2. P_1 lies on the circum-conic passing through G and X_{69}, the isotomic transform of the Euler line.
3. Ω_2 lies on the line X_6, X_{25}.

3.4 A selection of some remarkable cubics

The next table gives several examples of cubics of the pencil $\mathcal{F}(k)$ for some simple values of k. The cases $k = 0$ and $k = \infty$ are mentioned for completeness.

Notes :
(1) : the cubic is a central cubic.
(2) : the cubic is a $ps\mathcal{K}$.
(3) : the cubic is circular.
(4) : the cubic is a nodal (unicursal) cubic.

k	$\mathcal{S}(k)$	$\mathcal{C}(k)$	$\mathcal{P}_1(k)$	$\mathcal{P}_2(k)$
0	K003	K004	K002	K002
∞	$\mathcal{L}_\infty \cup \mathcal{J}$	$\mathcal{E} \cup \mathcal{O}$	K447 (3)	K446 (3)
1	K080 (1)	K080		K009 (4)
-1	K028 (2) (4)		K028 (4)	
$1/2$	K665			
$-1/2$	K026 (1) (2)			K026 (1)
$1/3$	K309			
$-1/3$	K581			
2		K443 (2)	K443	
-2	K525 (1)			K376
4		K426 (2)		K426
$4/3$		K047		
$-2/3$	K358			
$3/2$		K566		

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