Inscribed Cardioids and Eckart Cubics

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Abstract

We revisit Morley's inscribed cardioids and study the related Eckart cubics.

1 Cardioids

The cardioid is a very famous tricuspidal bicircular quartic of class 3 and the literature is quite considerable. Following Morley and others, we will focus on cardioids inscribed in the reference triangle ABC and those also tangent to a given line. We work with barycentric coordinates which is certainly not the most common approach for this type of problem but these are needed when we concentrate on the related cubics and other curves.

Note that any ABC center will be exclusively denoted by its ETC name to avoid any confusion : for example the circumcenter of ABC is X_3 and not (necessarily) O.

Let us denote by :

- $O = p : q : r$, the center of the cardioid $\mathcal{C}$,
- $K = u : v : w$, its cusp,
- $S = h_{O,-3}(K)$, its vertex,
- $\mathcal{A}$, its axis, the line passing through these three points,
- $P = h_{O,3}(K)$, the reflection of S about O,
- $\mathcal{B}$, its bitangent which is the perpendicular bisector of OP.

In the sequel, these points O, K are supposed to be finite and distinct.

A simple generation of $\mathcal{C}$ through envelopes of lines is the following. Let N be a variable point on the circle $C(O, P)$ such that $\angle(OP, ON) = t$. The line through P, parallel to the line ON meets this circle again at M. The line MN envelopes the cardioid and so does its perpendicular at M. Note that the two points of tangency lie on the parallel through K to ON.

It is then easy to give a parametric tangential equation of $\mathcal{C}$. With $T = \tan(t/2)$ to avoid trigonometry, an equation of the line MN can be expressed under the form :

$$T(3 - T^2)(p + q + r)T_0 + 8T^3 D_2(x + y + z) + 4\Delta(1 - 3T^2)(p + q + r)(u + v + w)D_0 = 0$$

where

- $D_0 = 0$ is the line OK i.e. the axis of $\mathcal{C}$ (obtained when $T = 0$),
- $T_0 = 0$ is the perpendicular at K to the line OK,
- Δ is the area of ABC,
- D_2 is a real positive number (which is zero when $O = K$ but here excluded) proportional to the squared distance OK. D_2 is actually a polynomial of global degree 2 in $p : q : r$, in $u : v : w$, in $a : b : c$ and symmetric with respect to these quantities.

The following table shows several special positions of the tangent MN according to the values of t and T and figure 1 below shows some of these tangents enveloping the cardioid.

Table 1: Special tangents to the cardioid

t	T	tangent MN
0	0	axis $\mathcal{A}$
$\pm\pi/3$	$\pm\sqrt{3}/3$	bitangent $\mathcal{B}$
$\pm\pi/2$	1	other tangents passing through P
$\pm 2\pi/3$	$\pm\sqrt{3}$	tangents parallel to $\mathcal{A}$
$\pm\pi$	$\pm\infty$	tangent at the vertex S

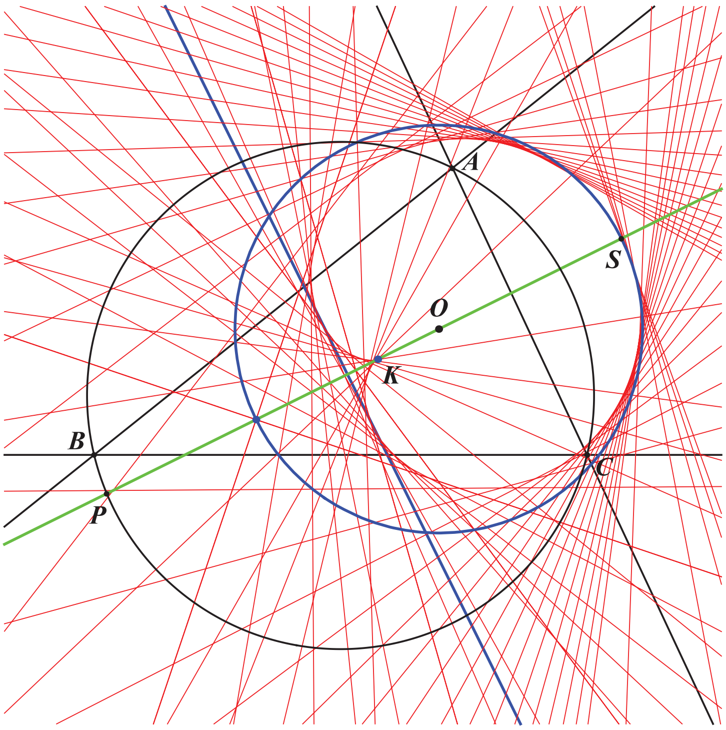

Figure 1: The cardioid as envelope of lines

2 Cardioids inscribed in triangle ABC

When we express that the line MN is one of the sidelines of ABC, we obtain two equations of global degree 2 in $p:q:r$ and in $u:v:w$, of degree 3 in T. It is convenient to replace one of them by their much simpler difference since it is linear in $p:q:r$ and in $u:v:w$. The resultant of these two latter polynomials (with respect to the variable T) gives a condition of global degree 3 in $p:q:r$ and in $u:v:w$.

Hence, a cardioid with center O and cusp K is inscribed in ABC if and only if three conditions E_A, E_B, E_C are fulfilled. See §4 for a geometric interpretation of these conditions.

In order to find the locus of the center O of these cardioids, we must eliminate the variables u, v, w between these conditions but recall that these are of degree 3. Any brute force computation is hopeless and we must find some additional particularities.

Let us then introduce the two quantities $T_1 = 27(p+q+r)D_2$ and $T_2 = 16\Delta^2(u+v+w)^3$ where D_2 is the real number defined above. It is very easy to verify that :

$$E_A = a^2u\,T_1 - p^3\,T_2, \quad E_B = b^2v\,T_1 - q^3\,T_2, \quad E_C = c^2w\,T_1 - r^3\,T_2.$$

The resolution gives :

$$u = \frac{p^3T_2}{a^2T_1}, \quad v = \frac{q^3T_2}{b^2T_1}, \quad w = \frac{r^3T_2}{c^2T_1},$$

showing immediately that the cusp K of the inscribed cardioid with center $O = p : q : r$ is the point :

$$\frac{p^3}{a^2} : \frac{q^3}{b^2} : \frac{r^3}{c^2}.$$

This point K can be seen as the pole of the isogonal conjugate O^* of O in the pencil of conics passing through O and the vertices of the anticevian triangle of O.

It is now sufficient to replace u, v, w by their values above in any of the equations E_A, E_B, E_C to obtain the locus of O : a curve of degree 9 which consists of three sets of three parallel lines as represented in figure 2 below.

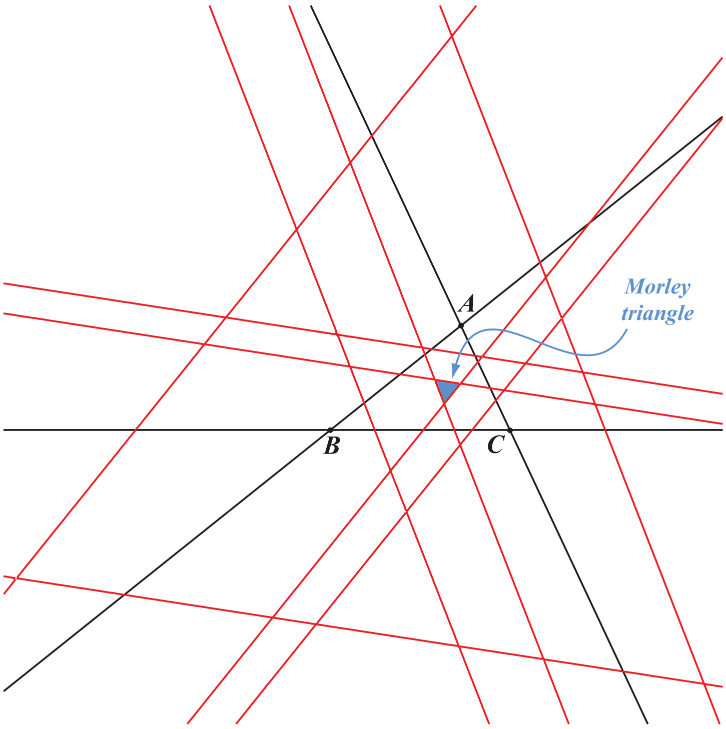

Figure 2: Locus of the centers of inscribed cardioids

The properties of these lines are well documented hence we shall not insist. Simply recall that they are related with the Morley triangles and that they are parallel to the asymptotes of the Kjp cubic K024 or equivalently perpendicular to those of the McCay cubic K003 we shall meet in the sequel.

Figure 3 shows one of these inscribed cardioids.

3 Cardioids inscribed in triangle ABC and tangent to a given line

Let L be the line $lx + my + nz = 0$ supposed not to be the line at infinity. We wish to find the inscribed cardioids of the previous paragraph which are also tangent to L.

We express that the tangent MN above is in fact the line L and we obtain three quite complicated conditions C_1, C_2, C_3 which are linear in l, m, n, of global degree 2 in $p : q : r$ and in $u : v : w$, of degree 3 in T.

We can verify that there is a linear combination L_3 of C_1, C_2 which is linear with respect to all the variables, of degree 2 in T but linear in T^2. It is then easy to find T^2 which we plug into C_3 to obtain L'_3 linear with respect to T.

The resultant E_L of the polynomials L_3, L'_3 is a polynomial of global degree 3 with respect to all the variables $p : q : r$, $u : v : w$, $l : m : n$ and symmetric in these variables. This is the

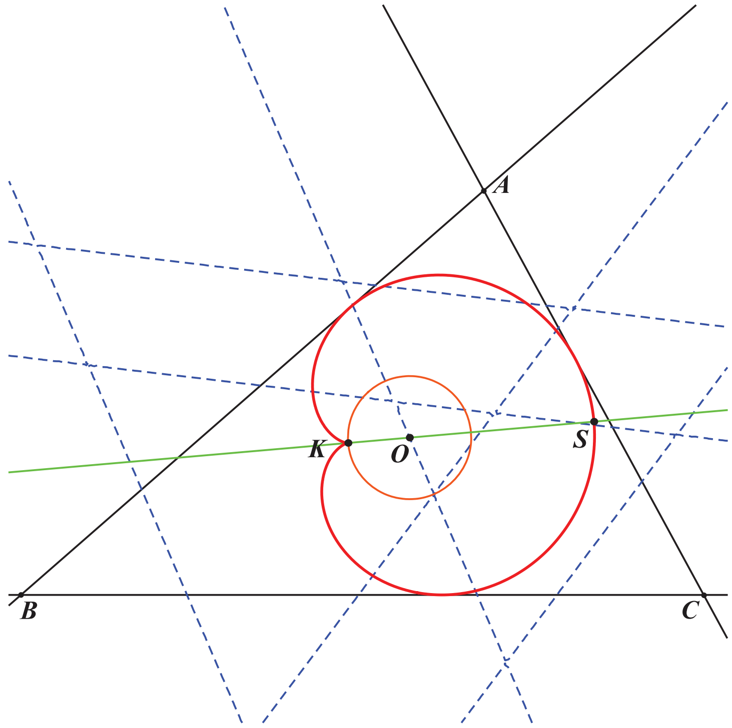

Figure 3: An inscribed cardioid

condition for which the cardioid is tangent to the line L.

Since E_L is quite huge (2376 terms), we must find a simple method to express that a cardioid is inscribed and tangent to L i.e. that the four conditions E_A, E_B, E_C, E_L are simultaneously fulfilled.

Let $E = k_1E_A + k_2E_B + k_3E_C + k_4E_L$ be a condition, supposed to be fulfilled, where all k_i are certain coefficients we shall determine. Naturally, if three of the four conditions above are fulfilled, the fourth is also fulfilled.

We write the system formed by the coefficients of E in u^3, v^3, w^3 and solve with respect to k_1, k_2, k_3. The values obtained are inserted in E and we obtain the product of two polynomials E_1, E_2 :

- E_1 is independent of u, v, w and it is symmetric of global degree 3 with respect to the variables p, q, r and l, m, n,

- E_2 is independent of l, m, n and it is symmetric of global degree 4 with respect to the variables p, q, r and 3 with respect to the variables u, v, w.

In other words, any point O lying on the 9 lines and on the cubic curve represented by E_1 is the center of a cardioid inscribed in ABC and tangent to L. This was discovered by Eckart Schmidt and this cubic curve is called the Eckart cubic $E(L)$ which we study in another section below. It easily follows that there are 27 such cardioids when the line L is given as far as it is not one of the 9 lines.

Figure 4 shows one of these 27 inscribed cardioids tangent to the line L.

On the other hand, the condition E_2 represents a cubic curve $\mathcal{K}_O$ with respect to O and a quartic curve $\mathcal{Q}_K$ with respect to K. It gives the locus of K (resp. O) for any suitably chosen O (resp. K) when the line L varies.

$\mathcal{K}_O$ is a circular circum-cubic with focus O. $\mathcal{Q}_K$ is also circular and even bicircular when $K = X_4$.

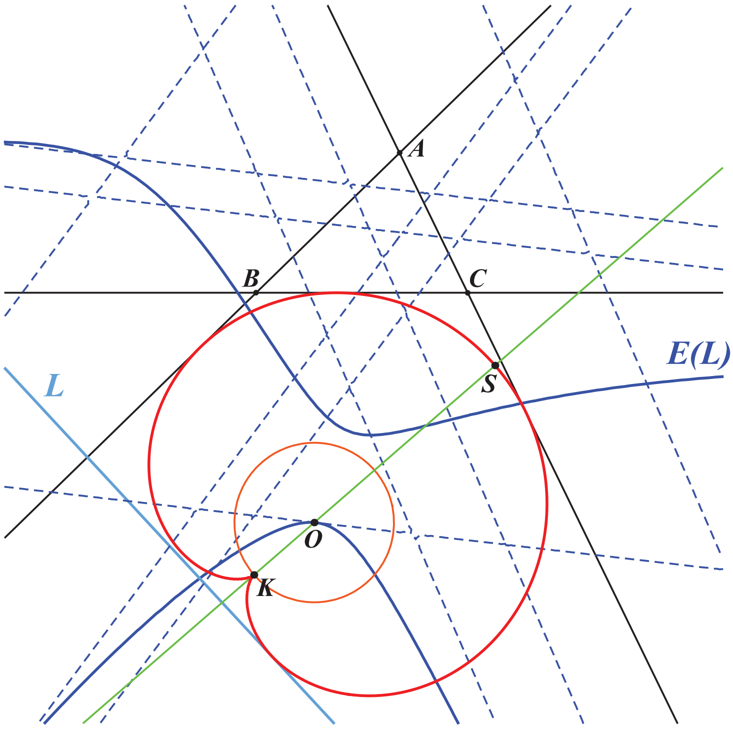

Figure 4: An inscribed cardioid tangent to the given line L

4 Geometric interpretations of the condition E_L

Recall that E_L is the condition for which a cardioid with center O, cusp K is tangent to the line L.

When one of these two points and the line L are given, we can identify the locus of the second point. We obtain two familiar cubics and both are sometime called trisectrices which is certainly not very surprising since a cardioid is also a trisectrix. The literature on this precise subject is quite plethoric. See [3] for example.

4.1 Locus of O when K and L are given

This locus is a Tschirnhausen cubic $T(L, K)$. If H is the projection of K on L, the node N is the homothetic of H under $h_{K,-2}$. It follows that, if we take $O = N$, then the corresponding cardioid is that bitangent to L and, if we take $O = N'$, then the corresponding cardioid is that with vertex H. See figure 5.

Construction of $T(L, K)$

We use the fact that $T(L, K)$ is the anti-pedal curve of the parabola (P) with focus K, directrix the perpendicular bisector of KH with respect to its focus.

Let E be a point on (P). The perpendicular L_E at E to KE envelopes $T(L, K)$. The normal at E to (P) meets the perpendicular bisector of KE at the center of the circle passing through K and tangent at E to (P). The antipode of K on this circle is the point of tangency of L_E. This latter point is the center O of a cardioid with cusp K tangent to L. When $O \neq N$, the line ON meets L at T, the contact of the cardioid with L.

4.2 Locus of K when O and L are given

This locus is a MacLaurin cubic $M(L, O)$ having L as real asymptote. If H is the projection of O on L, the node N is the homothetic of H under $h_{O,2/3}$. Note that, when $K = N$, the cardioid is bitangent to L. See figure 6.

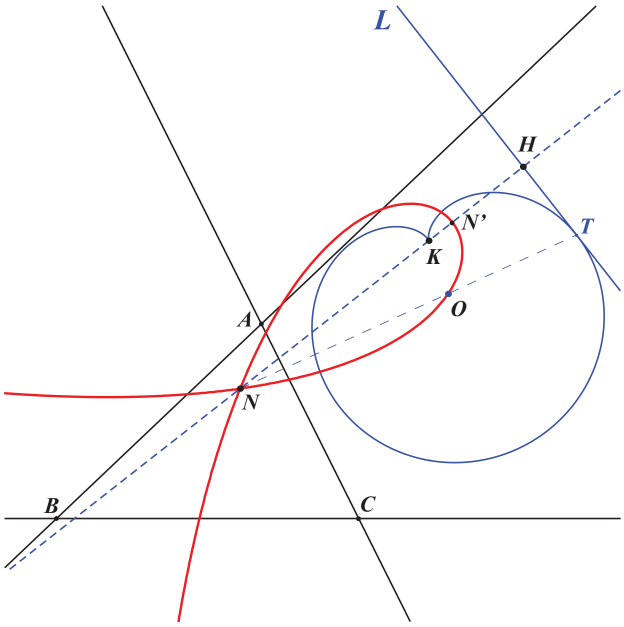

Figure 5: Tschirnhausen cubic and cardioid with given cusp K tangent to L

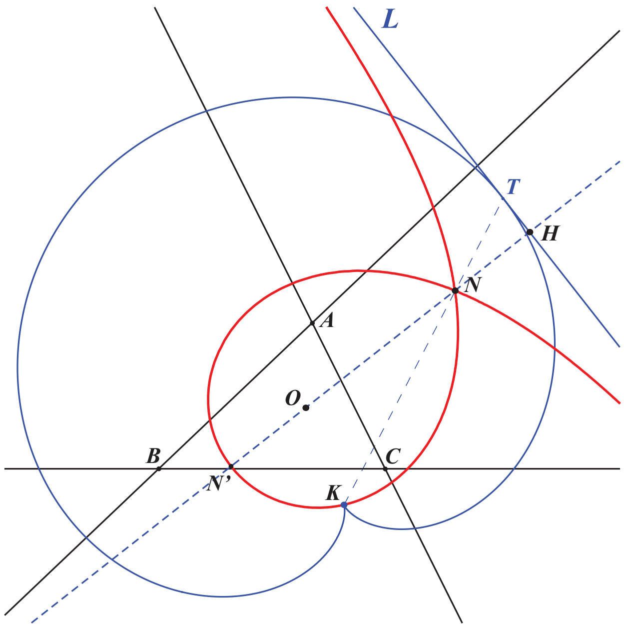

Figure 6: MacLaurin cubic and cardioid with given center O tangent to L

Construction of $M(L, O)$

$M(L, O)$ is the pedal curve of the parabola (P') with directrix the parallel at O to L and focus F', the reflection of N about O.

Let then E be a point on (P'). The projection of N on the tangent at E to (P') is the cusp K of the cardioid with center O tangent to L. If $K \neq N$, the line KN meets L at the contact of the cardioid with L.

4.3 Overview

The following figure 7 gathers together all these special curves and also shows one of the 27 cardioids related to L.

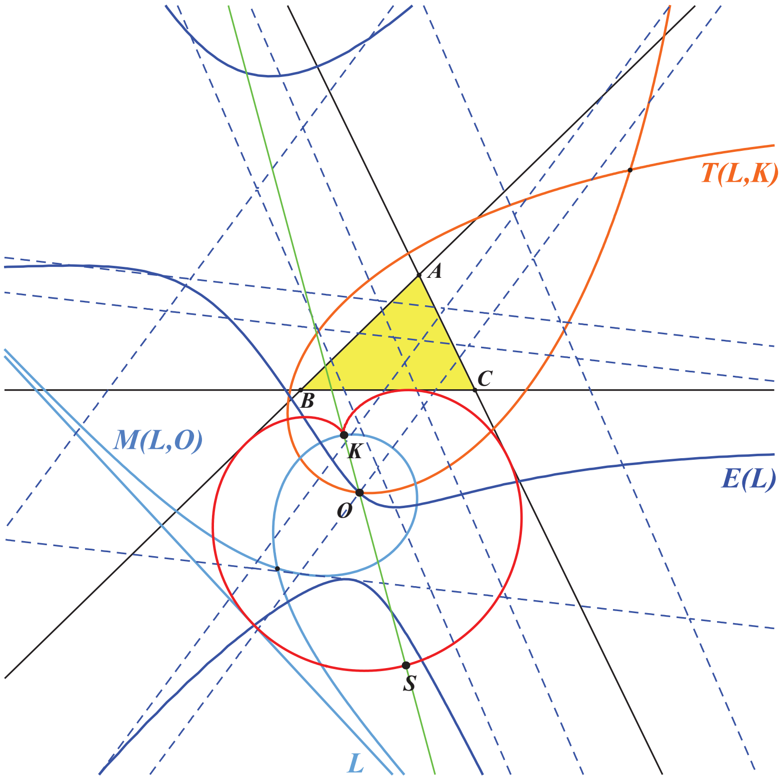

Figure 7: Overview of $E(L)$ with the Tschirnhausen and MacLaurin cubics

5 The Eckart cubic $E(L)$

Let L be the line with equation $lx + my + nz = 0$ supposed not to be the line at infinity i.e. such that the point $l : m : n$ is not the centroid X_2 of triangle ABC.

Let us recall that the center O of a cardioid inscribed in ABC and tangent to L must lie on $E(L)$ and on the nine lines we met above.

An equation of $E(L)$ is :

$$(lx + my + nz)^3 = \left(\sum_{\text{cyclic}} a^2(l - m)(l - n)\right)\left(\sum_{\text{cyclic}} l\,\frac{x^3}{a^2}\right).$$

The last factor represents a cubic $G(L)$ which cuts the line L at the same points as $E(L)$ and whose orthic line is L. The polar conics of each point of L with respect to $E(L)$ and $G(L)$ coincide. They are diagonal rectangular hyperbolas hence they must contain the in/excenters of ABC. The Hessian of $G(L)$ splits into the sidelines of ABC hence $G(L)$ meets each sideline at one real flex whose polar conic splits into the bisectors of the opposite vertex.

5.1 First properties of $E(L)$

The Laplacian of $E(L)$ identically vanishes hence $E(L)$ is a stelloidal cubic i.e. a cubic with three real asymptotes making 60° angles with one another and concurring at a point X which is the isogonal conjugate of the infinite point of the Newton line $N(L)$ of the quadrilateral formed by the sidelines of ABC and L. X is also the Miquel point of this quadrilateral. Naturally, this point lies on the circumcircle of ABC.

It follows that the polar conic in $E(L)$ of any point of the plane must be a rectangular hyperbola.

$E(L)$ meets its asymptotes at three finite collinear points S_1, S_2, S_3 lying on the satellite line $S(L)$ of the line at infinity. This line is the homothetic of $N(L)$ under $h_{X,2/3}$.

Eckart Schmidt observes that the asymptotes of $E(L)$ are actually parallel to the unique deltoid $H_3(L)$ inscribed in ABC and tangent to L. See figure 8.

Another description of these asymptotes is the following. With $\theta = \angle(N(L), L)$, consider the cubic $K(\theta)$ of the pencil of circum-stelloids generated by the McCay cubic K003 and the Kjp cubic K024 with equation :

$$2\Delta \cot\theta \sum_{\text{cyclic}} a^2\, x(c^2y^2 + b^2z^2) + \sum_{\text{cyclic}} a^2 S_A\, x(c^2y^2 - b^2z^2) = 0.$$

It is then easy to verify that $E(L)$ and $K(\theta)$ have the same points at infinity. After some computations, we find that the asymptotes of $E(L)$ are the parallels at X to those of K024 which have been rotated of the angle $\theta/3$. When $\theta = 0$ or $\theta = \pm 3\pi/2$, we obtain the two special cases examined in §9.4, §9.3 respectively. See below for the construction of these asymptotes.

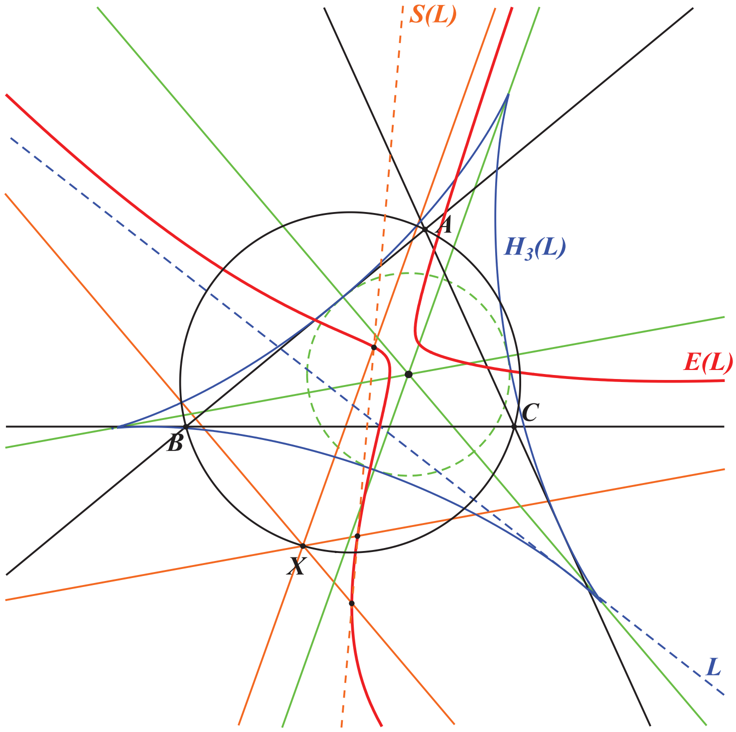

Figure 8: The Eckart cubic $E(L)$ and the deltoid $H_3(L)$

Construction of the deltoid $H_3(L)$

With θ as above, let M be a variable point on the circumcircle (O) of ABC. Under the rotation $R(M, \theta)$ with center M and angle θ, the parallels at M to BC, CA, AB are transformed into three other lines meeting BC, CA, AB at three collinear points on the line L_M. When M traverses (O), the line L_M envelopes the deltoid $H_3(L)$.

Furthermore, the center of $H_3(L)$ is the intersection of the perpendicular bisector of $X_3\,X_4$ and the perpendicular at X_3 to the image of the Euler line of ABC under $R(X_3, \theta)$.

5.2 Construction of the asymptotes of $E(L)$

First construction

This construction derives from that of the vertices of the circum-normal triangle and/or that of the asymptotes of the McCay cubic K003. It needs the trisection of an angle.

1. The Stammler hyperbola is the rectangular diagonal hyperbola passing through the in/excenters, X_3, X_6, etc, whose asymptotes are parallel to those of the Jerabek hyperbola and whose center is X_{110}. It is the polar conic of X_3 in the McCay cubic.

2. Its image under the homothety $h(X_3, 1/2)$ is another rectangular hyperbola passing through X_{110} and meeting the circumcircle again at the vertices N_1, N_2, N_3 of the circum-normal triangle. These latter points lie on K003 and the perpendiculars (resp. parallels) at X_2 to the sidelines of $N_1N_2N_3$ are the asymptotes of K003 (resp. K024). Obviously, this rectangular hyperbola contains X_3 and the four midpoints of the segments joining X_3 to the in/excenters.

3. Draw the parallels at X to these sidelines and rotate them around X of the angle $\theta/3$ where $\theta = \angle(N(L), L)$ to obtain the asymptotes of $E(L)$.

4. Finally, intersect these asymptotes with $S(L)$ to obtain the tangentials S_1, S_2, S_3 of the infinite points of $E(L)$. See figure 9.

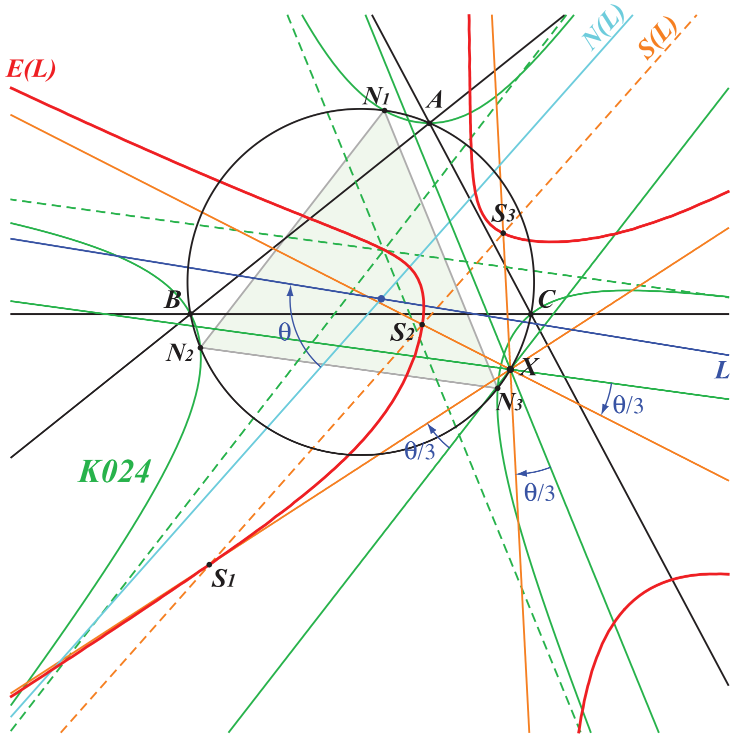

Figure 9: The Eckart cubic $E(L)$ and its asymptotes (construction 1)

Second construction

This construction needs to intersect conics.

1. Let Δ_0 be the line passing through the isogonal conjugates Y, X of the infinite points of L and $N(L)$ and let Δ_1 be its parallel at X_3.

2. Let $\mathcal{H}_1$ be the rectangular circum-hyperbola which is the isogonal transform of Δ_1.

3. Let $\mathcal{H}_2$ be the rectangular hyperbola passing through X_3 and X, whose center is the midpoint of these two points and whose asymptotes are parallel to those of $\mathcal{H}_1$.

4. $\mathcal{H}_2$ meets the circumcircle at X and three other points F_1, F_2, F_3 which are obviously the vertices of an equilateral triangle.

5. The asymptotes of $E(L)$ are the lines passing through X and each of the points F_1, F_2, F_3. See figure 10.

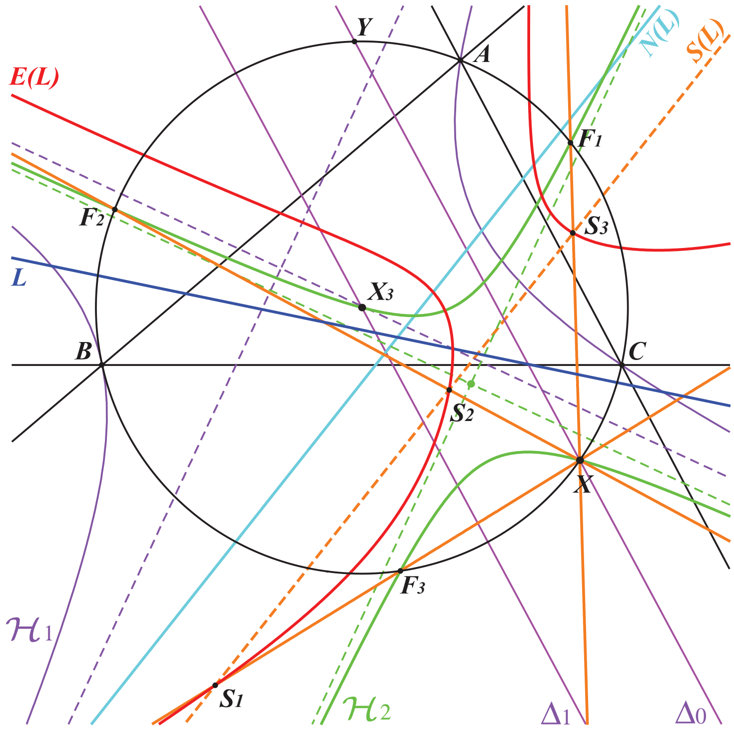

Figure 10: The Eckart cubic $E(L)$ and its asymptotes (construction 2)

5.3 Construction of $E(L)$

This construction uses the fact that the polar conics in $E(L)$ of any point M on L is a diagonal rectangular hyperbola. See §6 for further details about polar conics and figure 11.

1. Let M be a variable point on L and let $\mathcal{H}_1$ be its polar conic in $E(L)$. This is the hyperbola passing through the in/excenters of ABC whose center Ω_1 is the isogonal conjugate of the M-isoconjugate of the trilinear pole of L. Ω_1 obviously lies on the circumcircle of ABC.

2. The polar line $\mathcal{L}_1$ of M in $\mathcal{H}_1$ is transformed into the line $\mathcal{L}_2$ under $h(M, 2/3)$. Note that the envelope of $\mathcal{L}_1$ is the circum-conic whose perspector is the isogonal conjugate of the trilinear pole of L.

3. $\mathcal{H}_1$ and the decomposed conic which is the union of the line at infinity and $\mathcal{L}_2$ generate a pencil of homothetic rectangular hyperbolas which contains $\mathcal{H}_2$ whose center Ω_2 is the midpoint of $M\Omega_1$.

4. $\mathcal{H}_2$ meets the parallels at M to the asymptotes of $E(L)$ at six points of $E(L)$. See paragraph §5.2 for the construction of these asymptotes.

5. It follows that $E(L)$ is the locus of the intersections of $\mathcal{H}_2$ with these parallels when M traverses L.

6. Note that when M is one of the intersections U, V, W of L with the sidelines of ABC then $\mathcal{H}_1$ splits into the two bisectors at the corresponding vertex of ABC.

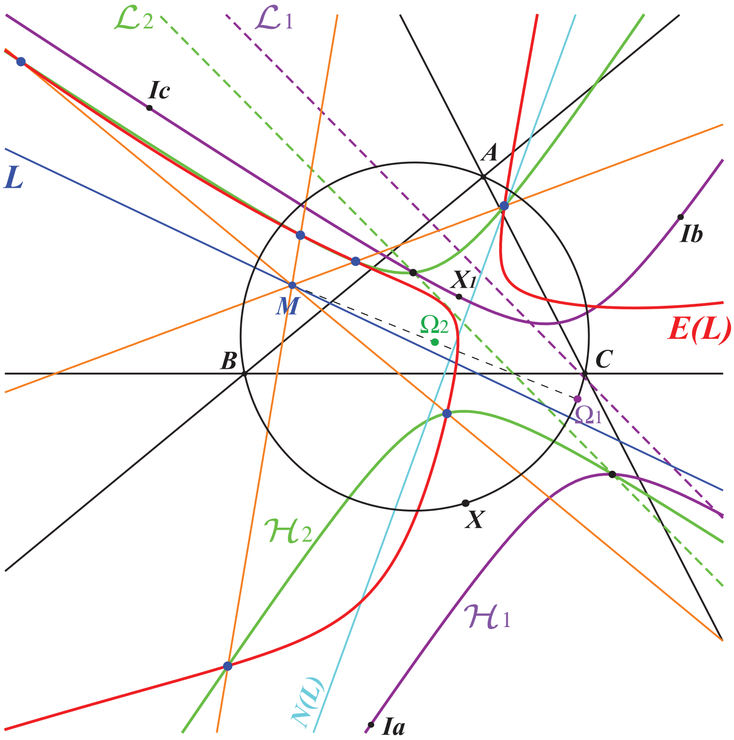

Figure 11: Construction of $E(L)$

5.4 The Hessian $H(L)$

The Hessian $H(L)$ of $E(L)$ is a focal circumcubic with singular focus X and orthic line $N(L)$. In other words, the polar conic of X is a circle and the polar conic of a point in $H(L)$ is a rectangular hyperbola if and only if this point lies on $N(L)$.

An equation of $H(L)$ is :

$$\sum_{\text{cyclic}} mn\, x(c^2y^2 + b^2z^2 + 2S_A\, yz) = 0.$$

$H(L)$ is actually an isogonal circular $n\mathcal{K}$ with root $R = mn : nl : lm$, the trilinear pole of L. This is the locus of the foci of the conics inscribed in ABC and having their center on $N(L)$. Recall that $H(L)$ meets the sidelines of triangle ABC at three points lying on the line L. See figure 12.

The nature of $H(L)$ has very important consequences in the sequel and depends of a certain discriminant Δ which is defined as follows.

5.5 Discriminant Δ and nature of the cubics $H(L)$ and $E(L)$

First, let us consider $\Delta_o = (-a+b+c)mn + (-b+c+a)nl + (-c+a+b)lm$, which is zero if and only if R lies on the trilinear polar of the Gergonne point X_7 or equivalently L is tangent to the incircle. Furthermore, $\Delta_o < 0$ if and only if L is secant to the incircle and $\Delta_o > 0$ if and only if L is exterior to the incircle. In other words, Δ_o is a positive multiple of the difference between the distance from X_1 to L and the radius of the incircle.

Three other quantities Δ_a, Δ_b, Δ_c are defined likewise by extraversion and correspond to the excircles. Then, $\Delta = \Delta_o \times \Delta_a \times \Delta_b \times \Delta_c$ is their product.

Each quantity Δ_i is related to the trilinear polar of X_7 or one of its extraversions and to the incircle or one excircle. These four trilinear polars — denoted on figure 13 below by $(X7o)$ for that of X_7 and $(X7a)$, $(X7b)$, $(X7c)$ for its extraversions — determine eleven regions and, in each region, the signs of Δ_o, Δ_a, Δ_b, Δ_c and therefore Δ are known. Recall that $\Delta_i = 0$ (and

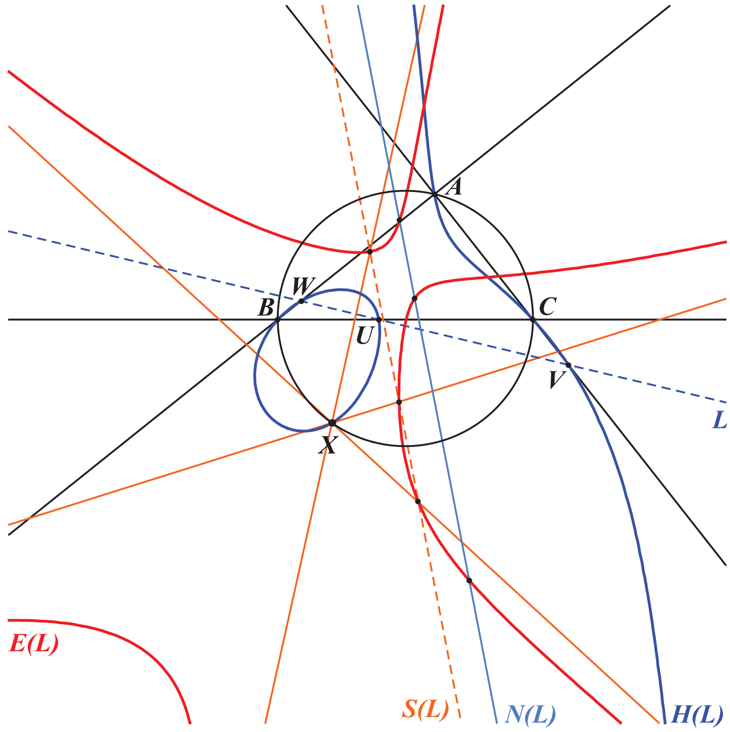

Figure 12: The Eckart cubic $E(L)$ and its Hessian $H(L)$

then $\Delta = 0$) when R lies on at least one of the trilinear polars. Note that each line $(X7i)$ splits the plane into two half-planes and Δ_i is positive in that containing the centroid X_2 of ABC.

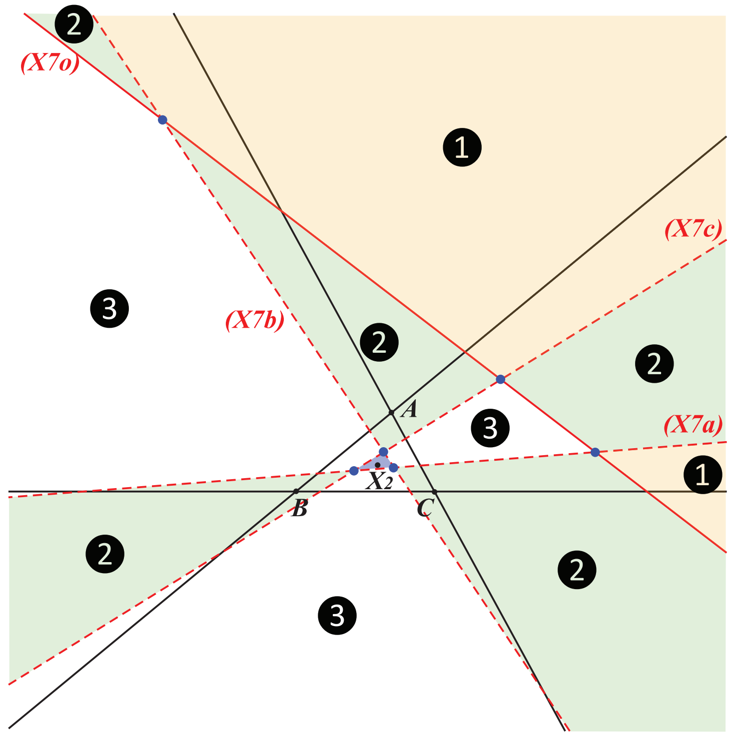

Figure 13: Eleven regions

There are in fact four types of regions depending of the number of positive quantities Δ_i i.e. the number N of in/excircles that do not cut the line L :

- the (small) blue region containing X_2 corresponding to $N = 4$: L is outside all four circles,
- the white regions corresponding to $N = 3$: L intersects one circle,
- the green regions corresponding to $N = 2$: L intersects two circles,
- the orange regions corresponding to $N = 1$: L intersects three circles.

From this we find a classification of the cubics $H(L)$ according to the position of R in the plane and consequently the number of real intersections of $H(L)$ and $N(L)$ and, at last, the number of real intersections of $E(L)$ and $N(L)$.

Indeed, when $N(L)$ is parametrized with a point $M(t)$ and when $M(t)$ is plugged into the equations of $H(L)$ and $E(L)$, we obtain two third degree polynomials $P(t)$ and $Q(t)$ with the following properties :

- $P(t)$ is the product of a linear factor – which gives the infinite point of $N(L)$ lying on $H(L)$ — and a quadratic factor whose discriminant has the opposite sign of Δ.
- the discriminant of $Q(t)$ has the sign of Δ.

Thus, $H(L)$ and $E(L)$ are singular for the same choice of R which is not surprising since the Hessian of a singular cubic must contain the singular point.

Recall that when R is one of the six (blue) points, the intersection of two trilinear polars, $H(L)$ splits into a line and a circle, and when R lies on only one trilinear polar, $H(L)$ is a nodal cubic and actually a strophoid.

This is detailed and illustrated in the next paragraphs but we first gather together the different possible configurations in Table 2.

Table 2: Discriminant Δ and consequences

	$\Delta > 0$	$\Delta < 0$	$\Delta = 0$
N	even, 2 or 4	odd, 1 or 3	
$H(L) \cap N(L)$	2 compl. conj.	2 real distinct	1 real double
$H(L)$ is	bipartite	unipartite	singular
$E(L) \cap N(L)$	3 real distinct	1 real, 2 compl. conj.	1 real, 1 real double

5.6 Decomposed cubics $H(L)$ and $E(L)$

There are 6 decomposed cubics of each type, each containing one of the 6 bisectors of ABC which is the line $N(L)$.

$H(L)$ also contains the circle passing through the two other vertices of ABC and the two in/excenters on the bisector which are the end points of one diameter.

$E(L)$ is the hyperbola with excentricity 2, passing through these same in/excenters and having the same center as the circle namely the second intersection of the bisector with the circumcircle of ABC. See figure 14 where $N(L)$ is the external bisector at A.

5.7 Nodal cubics $H(L)$ and $E(L)$

Each point R lying on one (and only one) of the four trilinear polars above gives a line L tangent to one in/excircle and two nodal cubics $H(L)$ and $E(L)$ with common node at the relative in/excenter I_x and also comon nodal tangents.

$H(L)$ is a strophoid $c\mathcal{K}(\#I_x, R)$ with focus F on the circumcircle and $E(L)$ is a nodal stelloid with radial center F. See figure 15 where R is a point on the trilinear polar $(X7b)$ hence the

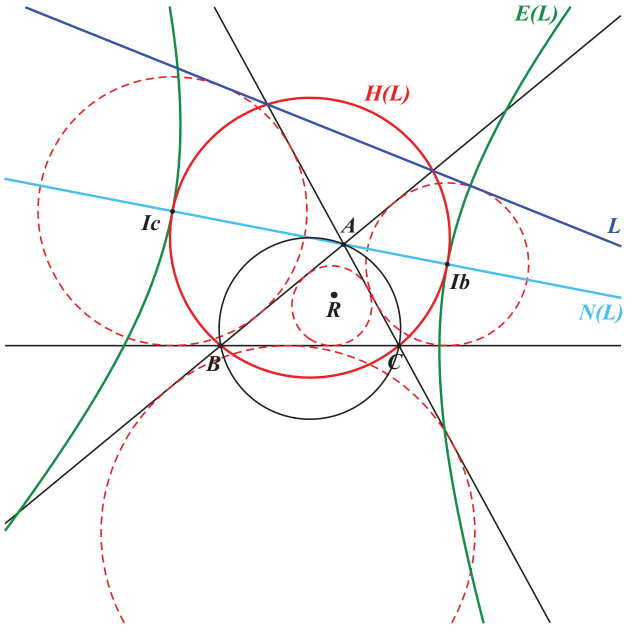

Figure 14: Decomposed cubics $H(L)$ and $E(L)$: L is tangent to two in/excircles

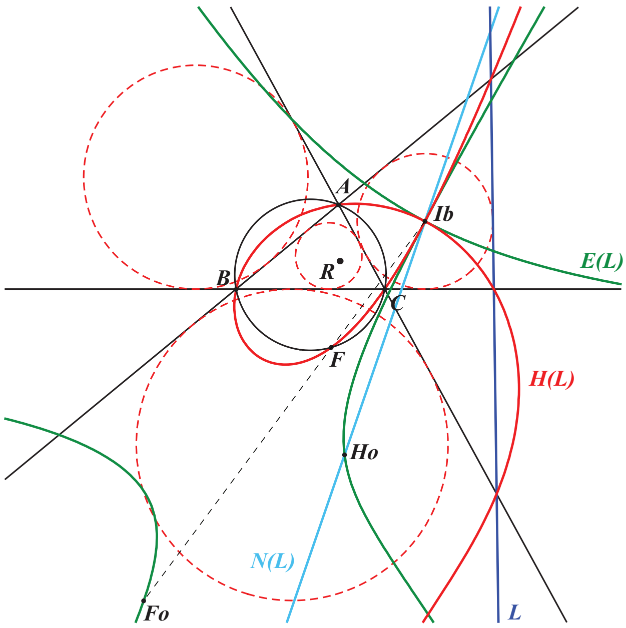

Figure 15: Nodal cubics $H(L)$ and $E(L)$: L is tangent to one in/excircle

node is the excenter I_b, L is tangent to the $B-$excircle. The line $N(L)$ must contain I_b and therefore meets $E(L)$ at another real point H_o.

Simultaneous construction of the nodal cubics $H(L)$ and $E(L)$

We suppose that the node is the incenter X_1 of ABC and this will be easily adapted for the excenters.

Let $N(L)$ be a line through X_1 and F the isogonal conjugate of its infinite point.

1. Construct the rectangular hyperbola (H) passing through X_1, F and the inverses A', B', C' of A, B, C in the circle $C(X_1, F)$.

2. The inverse of (H) is $H(L)$.

3. The parallels to $N(L)$ passing through the reflections of A, B, C about X_1 meet the sidelines of ABC at U, V, W lying on the line L.

4. Let F_o be the homothetic of F under $h(X_1, 3)$ or the homothetic of X_1 under $h(F, -2)$. The perpendicular bisector (D) of X_1F_o meets $N(L)$ at H_o. These points F_o and H_o lie on $E(L)$.

5. Let G_o be a variable point on (D). Construct :

   – the circle (C_o) with center G_o passing through X_1 and Fo,

   – the rectangular hyperbola (H_o) passing through X_1, H_o, G_o whose center is the midpoint of X_1G_o.

6. (C_o) and (H_o) meet at X_1 and three other points P_1, P_2, P_3 which lie on $E(L)$. These are the vertices of an equilateral triangle since X_1 (counted twice) and F_o are the pivots of the stelloid $E(L)$.

7. $C(X_1, F)$ and (H) meet at F and three other points E_1, E_2, E_3 which are the vertices of an equilateral triangle. The lines FE_i are the asymptotes of $E(L)$. They meet $E(L)$ again at three (collinear) points lying on the satellite line of the line at infinity namely the homothetic of $N(L)$ under $h(F, 2/3)$.

Alternative construction of certain elements

Instead of constructing points on $E(L)$ by groups of three such as P_1, P_2, P_3, it is possible (and more simple) to get pairs of points on $H(L)$ (this is well known) and also on $E(L)$.

With $N(L)$ and F as above,

1. Let L_F be a variable line through F meeting $N(L)$ at ω. The circle $C(\omega, X_1)$ meets L_F at two points N_1, N_2 on $H(L)$.

2. Let L'_F be the perpendicular bisector of FF_o (with F_o as above). L'_F meets L_F at a point ω'. The line $F_o\omega'$ meets X_1N_1, X_1N_2 at M_1, M_2 respectively which are two points on $E(L)$ obviously collinear with F_o.

Note that the line L'_F passes through the common real point of inflexion of the two cubics.

Remark : For any M on $E(L)$, we have $2\angle(X_1H_o, X_1M) + \angle(F_oH_o, F_oM) = 0\,(\text{mod.}\pi)$. This is a classical property of such nodal stelloids.

Note also that, for a given line L, this type of cubic has only one group of pivots containing a "double pivot" since the node X_1 must be this double pivot, F must be the radial center and, at the same time, the centroid on any group of pivots hence the third pivot must be F_o.

Construction of groups of pivots on a nodal cubic $E(L)$ with node X_1

First construct any point P_1 on $E(L)$ as above and let P'_1 be its image under the homothety with center F and ratio $-1/2$. Consider the ellipse $(\mathcal{E})$ with center F, focus X_1 that passes through P'_1. Its tangent at P'_1 meets its two tangents drawn through P_1 at two points P_2, P_3 which are the requested pivots associated with P_1. This ellipse $(\mathcal{E})$ is obviously the Steiner inscribed ellipse of the triangle formed by these three pivots P_1, P_2, P_3. Recall that the circumcircle of this triangle meets $E(L)$ again at the vertices of an equilateral triangle.

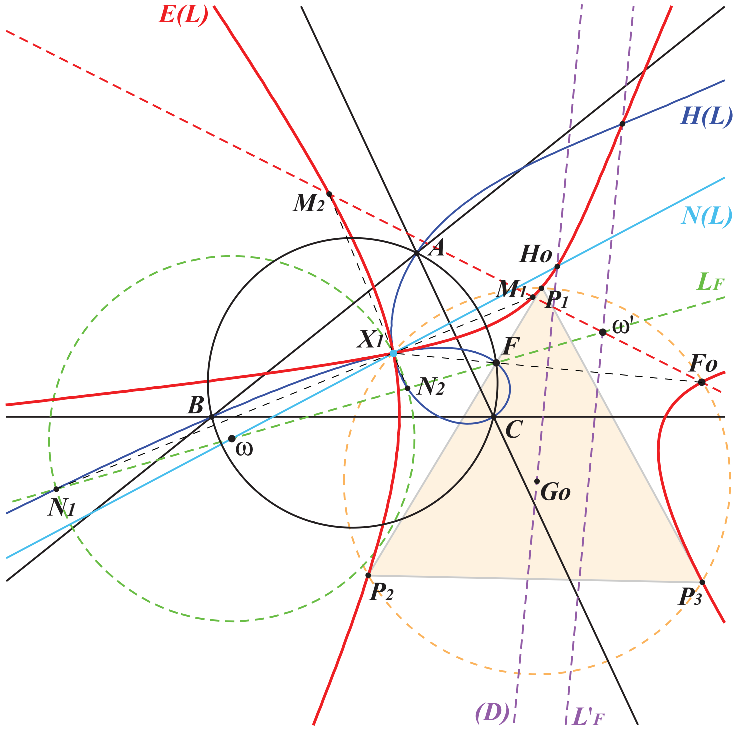

Figure 16: Nodal cubics $H(L)$ and $E(L)$: a construction

5.8 Bipartite or unipartite cubic $H(L)$ and $E(L)$

These are the cubics obtained when $\Delta \neq 0$. They are non-singular cubics of class 6.

- $H(L)$ is a bipartite focal cubic when L intersects an even number of in/excircles. It has an oval part and an infinite branch containing the three real points of inflexion. In this case, $H(L)$ meets $N(L)$ at one real point (which is at infinity) and two complex conjugate finite points whose midpoint Ω (constructed below) is real. $E(L)$ meets $N(L)$ at three real points Ω_1, Ω_2, Ω_3. See figure 18.

- $H(L)$ is an unipartite focal cubic when L intersects an odd number of in/excircles. It has only one infinite branch but it is not an unicursal curve. $H(L)$ meets $N(L)$ at one real point (which is at infinity) and two other real finite points with midpoint Ω. $E(L)$ meets $N(L)$ at one real point Ω_o and two other complex conjugate finite points. See figure 17.

Construction of the cubics $H(L)$

Let U, V, W be the traces of L on the sidelines of triangle ABC. These points lie on $H(L)$. Let A' be the intersection of AF with the parallel at U to $N(L)$ and A'' the intersection of UF with the parallel at A to $N(L)$. Four other points B', B'' and C', C'' are defined likewise and all six lie on $H(L)$.

This also gives the polar conic of the infinite point of $N(L)$ in $H(L)$ since it must contain the midpoint of any segment parallel to $N(L)$ whose end points lie on $H(L)$ such as $A'U$, AA'', etc. This conic is a rectangular hyperbola whose center is the reflection of F about $N(L)$. It contains Ω. One asymptote is that of $H(L)$ and the other is its perpendicular at F.

The circles with diameters AA', BB', CC', UA'', VB'', WC'' are in a same pencil $\mathcal{F}$ whose radical axis $\mathcal{R}$ is the perpendicular at Ω to $N(L)$. Note that $\mathcal{R}$ meets the parallel at F (the singular focus) to $N(L)$ at Y, a point on $H(L)$ whose isogonal conjugate X is the intersection of $H(L)$ with its real asymptote.

Now, if M is a variable point on $N(L)$, the line FM meets the circle of $\mathcal{F}$ whose center is M at two points N_1, N_2 on the cubic $H(L)$.

- When $H(L)$ is unipartite, all the circles of $\mathcal{F}$ are orthogonal to a real circle with center Ω, meeting $N(L)$ at R_1, R_2 which are the real foci of the inscribed conic with center Ω and tangent to L hence whose focal axis is $N(L)$. Note that the circle through F, R_1, R_2 is the polar conic of F in $N(L)$. See figure 17.
- When $H(L)$ is bipartite, all the circles of $\mathcal{F}$ pass through two real points B_1, B_2 which lie on the circle with diameter FX and obviously on $\mathcal{R}$. See figure 18.

In both cases, the centers of anallagmaty E_i of $H(L)$ are the intersections of the angle bisectors of FX, FY with the polar conic of the infinite point of $N(L)$ in $H(L)$. In the first case (figure 17), two only are real and, in the second case (figure 18), all four are real which means that $H(L)$ is an isogonal pivotal cubic with respect to the triangle FB_1B_2 with pivot the infinite point of $N(L)$ and with isopivot X.

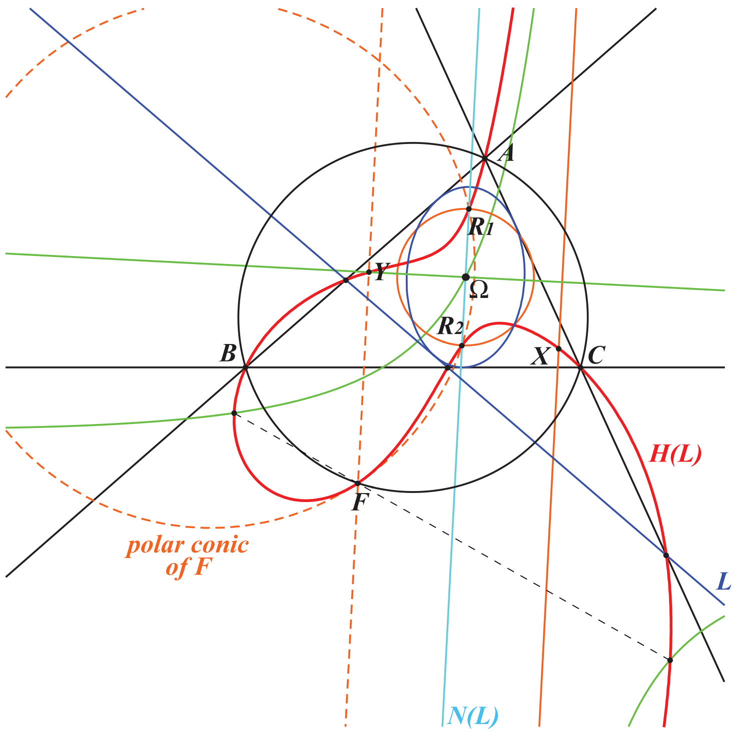

Figure 17: $H(L)$ is unipartite

6 Polar conics in $E(L)$ and $H(L)$

Recall that the polar conic of F in $H(L)$ is a circle, obviously passing through F, whose center is the intersection of the perpendicular at F to FX and the perpendicular at Y to $N(L)$.

Recall also that the polar conic of any point in $E(L)$ is a rectangular hyperbola and that the polar conic of a point in $H(L)$ is a rectangular hyperbola if and only if this point lies on $N(L)$, the orthic line of $H(L)$.

The polar conics in $E(L)$ of the points U, V, W (the intersections of L and the sidelines of ABC) are the pairs of bisectors at A, B, C hence the poles of the line L are the in/excenters of ABC. It follows that the polar conic of every point on L is a diagonal rectangular hyperbola whose center obviously lies on the circumcircle of ABC.

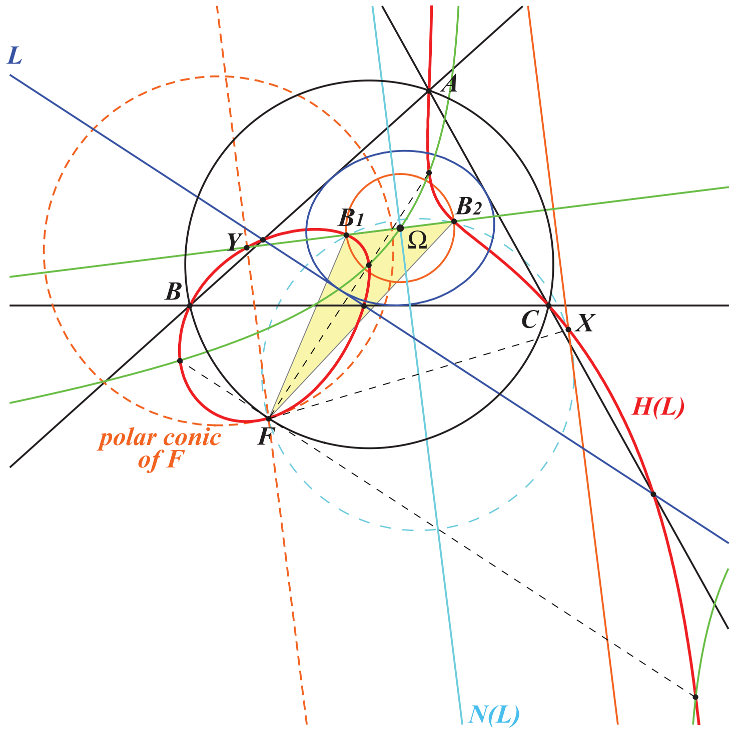

Figure 18: $H(L)$ is bipartite

The polar conic of the infinite point of $N(L)$ in $E(L)$ must split into two perpendicular lines L_1, L_2 since this point lies on $H(L)$. It appears that these perpendicular lines are precisely the angle bisectors of FX, FY thus passing through the centers of anallagmaty E_i of $H(L)$ seen above. It follows that the tangents at the common points of these angle bisectors and both cubics $E(L)$ and $H(L)$ are parallel to $N(L)$.

More generally, the polar conic of any infinite point is a rectangular hyperbola with center F and these hyperbolas form a pencil with four common points lying on L_1, L_2 above. Since these hyperbolas are concentric, two points are real and two are complex conjugates. Now, if the infinite point is chosen to be that of any perpendicular to $N(L)$ then the rectangular hyperbola must be symmetric about any of the two lines L_1, L_2. It follows that the two real points are the vertices F_1, F_2 of the hyperbola. See figure 19.

The circle Γ with diameter F_1F_2 and the lines L_1, L_2 are associated to a natural conjugation on $H(L)$. For any point M in the plane, let $\Phi(M)$ be the center of the polar conic of M in $E(L)$. $\Phi(M)$ is the image of M under the (commutative) product of an inversion in Γ and a reflection in the line F_1F_2.

Now, M on $H(L)$ is carried to another point also on $H(L)$ which is the usual conjugation on focal cubics : the parallel at M to the asymptote meets $H(L)$ again at M' and then $\Phi(M)$ is the third point of $H(L)$ on the line FM'.

It is interesting to note that this transformation Φ swaps $N(L)$ and the polar conic of F in $H(L)$ since it gives a construction of F_1 and F_2.

Construction of F_1, F_2 and Γ

- Construct the Newton line $N(L)$ and F, the isogonal conjugate of its infinite point.
- Construct X (intersection of $H(L)$ and its asymptote, see above) and the circle with diameter FX.
- This circle meets $N(L)$ at two points. The lines through F and these points are the lines L_1, L_2.

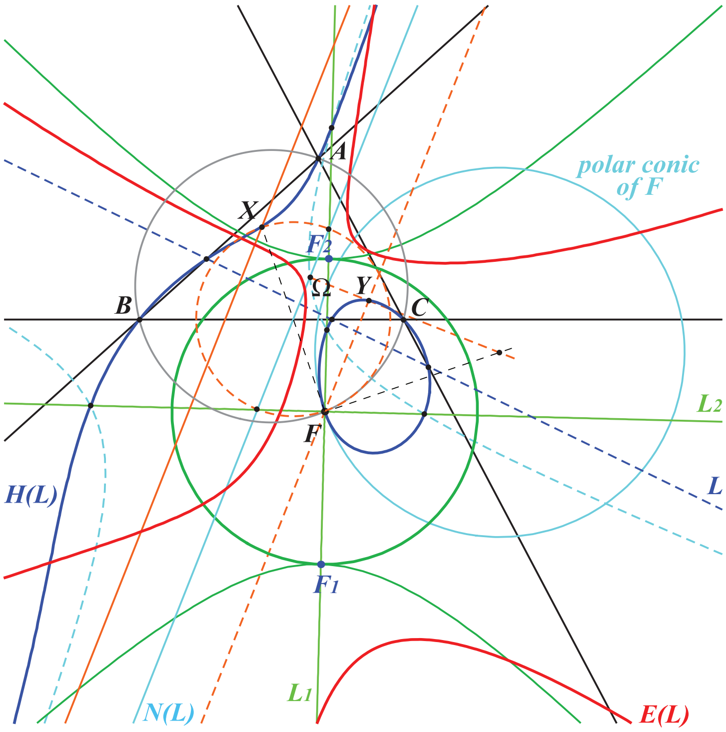

Figure 19: Pencil of polar conics

- Draw the circle which is the polar conic of F in $H(L)$ as above and reflect this latter circle about L_1 or L_2.
- One of these circles meets $N(L)$ at two real points. Draw the circle Γ with center F passing through these two points intersecting the chosen line L_1 or L_2 at the requested points F_1, F_2.

7 Groups of pivots on a non singular cubic $E(L)$

We have already seen the construction of groups of pivots on a nodal cubic $E(L)$ and we now wish to realize the same thing on any other cubic $E(L)$.

It is known that the polar of any order in any stelloid is also a stelloid and, which is the case here, the polar conic of any point in $E(L)$ is a rectangular hyperbola. Each stelloid is associated to a group of pivots : a group contains 3 points for a stelloidal cubic and 2 antipodal points for a rectangular hyperbola. The isobarycenter must be the same for all groups and any pivot must lie on the curve.

If a group of pivots P_1, P_2, P_3 is known on the cubic then the centroid of $P_1P_2P_3$ must be the radial center X of the stelloid. Furthermore, the polar conics of the point at infinity are centered at X also then each must have a group of two pivots which are the (real) foci of the inscribed Steiner ellipse of triangle $P_1P_2P_3$.

These foci are precisely the points F_1, F_2 we defined above. See [4, 7, 11, 14, 15, 17] for justifications.

In other words, if M, N are two points on the stelloid with group of pivots P_1, P_2, P_3 then

$$\angle(P_1M, P_1N) + \angle(P_2M, P_2N) + \angle(P_3M, P_3N) = 0 \text{ (mod. } \pi).$$

Naturally, this same equality holds with a rectangular hyperbola but with only two pivots. This is a well known property of rectangular hyperbolas.

If a stelloid $E(L)$ is not nodal, there are infinitely many groups of three distinct pivots on the curve and it is sufficient to know one of them – say P_1 – to obtain the others P_2, P_3.

Construction of groups of pivots on $E(L)$

Let us consider that a point P_1 is known on $E(L)$. For example, P_1 can be one of the points S_1, S_2, S_3 of §5.1.

- Construct F_1, F_2 as above with midpoint X. Recall that X is here the singular focus of $H(L)$ and the radial center of the stelloid.
- Construct Q_1, the homothetic of P_1 under $h(X, -1/2)$.
- Draw the external bisector of Q_1F_1, Q_1F_2 (this is the tangent at Q_1 to the Steiner ellipse now entirely defined).
- Draw the tangents through P_1 to this ellipse.
- These tangents meet the bisector at P_2, P_3.

Remarks :
– The circumcircle of triangle $P_1P_2P_3$ meets $E(L)$ at three other points which are the vertices of an equilateral triangle. See figure 20.
– Since each of the points S_1, S_2, S_3 corresponds to two other points on $E(L)$, we already know twelve points on $E(L)$ including its points at infinity (see §5.1).

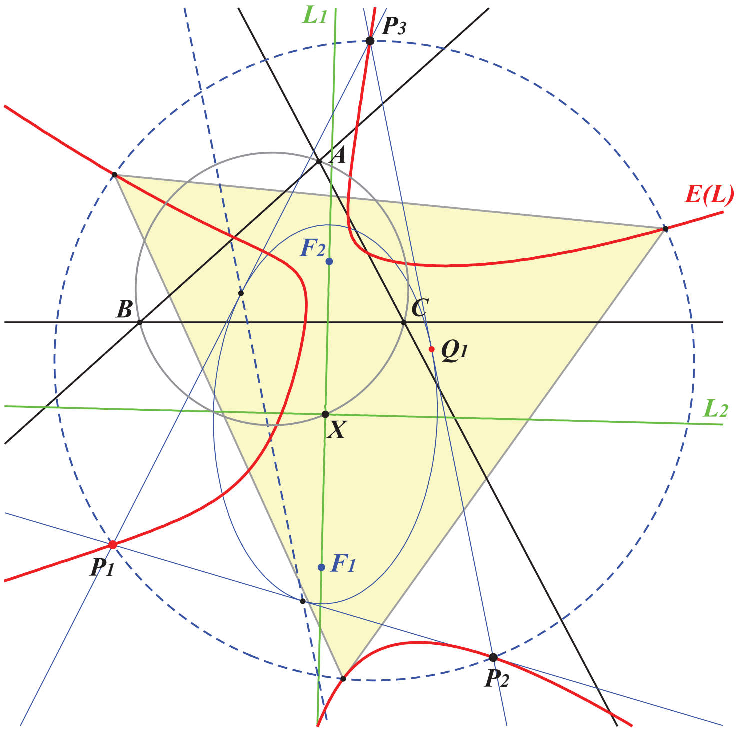

Figure 20: Group of pivots P_1, P_2, P_3 on $E(L)$

It follows that $E(L)$ is a stelloidal circumcubic in infinitely many triangles $P_1P_2P_3$ but certainly not in general an isocubic similar to the McCay cubic K003 (isogonal $p\mathcal{K}$) or to the Kjp cubic K024 (isogonal $n\mathcal{K}_0$).
This is examined in the next sections.

8 Prehessians of $E(L)$ and consequences

8.1 Introduction

Recall that $E(L)$ and its Hessian $H(L)$ define a pencil $\mathcal{F}(L)$ of cubics all sharing the same points of inflexion. Any member of this pencil can be written under the form $F(t) = (1 - t)\, E(L) + t\, H(L)$ where t is a real number. The Hessian $H(t)$ of $F(t)$ is also a member of the pencil.

A cubic is said to be a prehessian of $E(L)$ if and only if its Hessian is $E(L)$. It is then sufficient to identify $H(t)$ and $E(L)$ but, since these have already nine common (inflexion) points, it is enough to write that their difference vanishes in any other point and this gives a polynomial of degree 3 in t.

The number of real prehessians is then the numbers of real roots.

After a quite tedious computation of the discriminant of this latter polynomial we obtain it has the opposite sign of the discriminant Δ of paragraph 5.5.

Gathering together the results above, we obtain :

- When $H(L)$ is bipartite ($\Delta > 0$), $E(L)$ meets $N(L)$ at three real points Ω_1, Ω_2, Ω_3 and it has only one real prehessian Π_0.
- When $H(L)$ is unipartite ($\Delta < 0$), $E(L)$ meets $N(L)$ at one real point Ω_0 and it has three real prehessians Π_1, Π_2, Π_3.
- When $H(L)$ is singular ($\Delta = 0$), $E(L)$ is also singular and cannot be of class 6.

We can now identify two types of cubics $E(L)$ corresponding to the first two situations above according to the known properties of the McCay cubic and the Kjp cubic. This will give the results of the next two paragraphs.

8.2 Cubics $E(L)$ of the McCay type

Theorem 1 *The following assertions are equivalent :*

1. *L cuts an odd number in/excircles (1 or 3).*
2. *$H(L)$ is unipartite and meets $N(L)$ at three real points (one is at infinity).*
3. *$E(L)$ meets $N(L)$ at only one real point Ω_0.*
4. *$E(L)$ has three real prehessians Π_1, Π_2, Π_3.*
5. *$E(L)$ is the McCay cubic of one and only one triangle $A'B'C'$ whose circumcenter is Ω_0 and whose orthocenter H_0 is the tangential of Ω_0 in $E(L)$.*

The vertices A', B', C' are the centers of the polar conics of Ω_0 with respect to Π_1, Π_2, Π_3. These polar conics are rectangular decomposed hyperbolas which are the bisectors of $A'B'C'$ intersecting at the in/excenters of $A'B'C'$ which lie on $E(L)$.

The Euler line of $A'B'C'$ is the tangent at Ω_0 to $E(L)$. It meets $H(L)$ at two points which are antipodal on the circumcircle of $A'B'C'$. See figure 21.

8.3 Cubics $E(L)$ of the Kjp type

Theorem 2 *The following assertions are equivalent :*

1. *L cuts an even number in/excircles (0 or 2).*
2. *$H(L)$ is bipartite and meets $N(L)$ at one real point (at infinity).*
3. *$E(L)$ meets $N(L)$ at three real points Ω_1, Ω_2, Ω_3.*

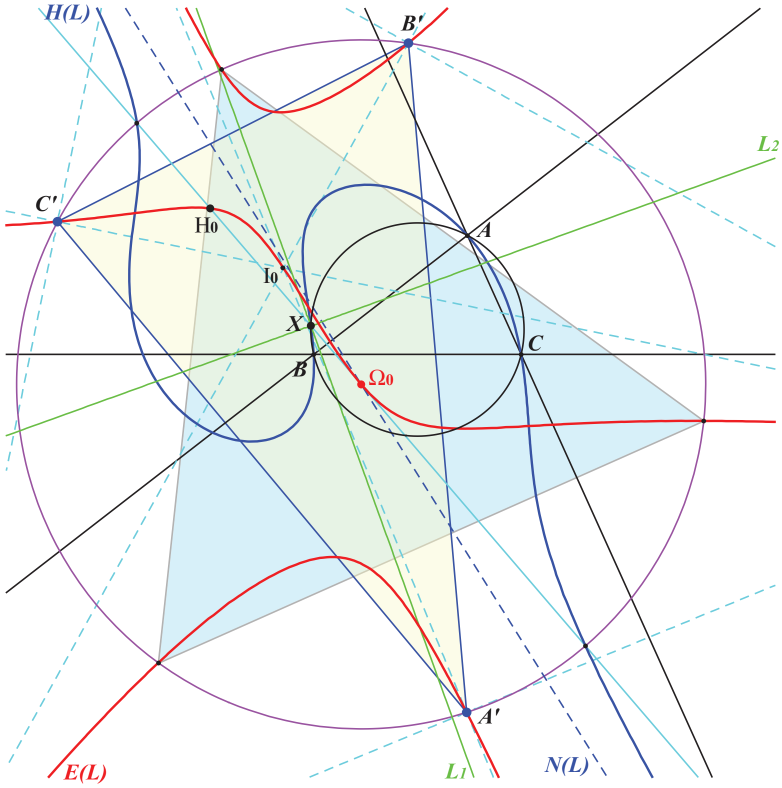

Figure 21: Cubic $E(L)$ of the McCay type

4. *$E(L)$ has only one real prehessian Π_0.*

5. *$E(L)$ is the Kjp cubic of one and only one triangle $A'B'C'$.*

The vertices A', B', C' are the centers of the polar conics of Ω_1, Ω_2, Ω_3 with respect to Π_0. These polar conics are again the bisectors of $A'B'C'$.

$E(L)$ is tritangent to the poloconic of $N(L)$ with respect to Π_0. This is the Steiner circum-ellipse of $A'B'C'$.

The points Ω_1, Ω_2, Ω_3 are the centers of the Apollonius circles of triangle $A'B'C'$. See figure 22.

9 Special cases and other properties

The following paragraphs present various types of cubics according to one special feature.

9.1 Cubics $E(L)$ having the same radial center

Let $E(L')$ be the cubic associated to the line L' with equation $l'x + m'y + n'z = 0$. The cubics $E(L)$ and $E(L')$ have the same radial center X if and only if L and L' are tangent to the same inscribed parabola $P(L)$, that of focus X.

9.2 Cubics $E(L)$ having the same asymptotic directions

With the same notations as above, the cubics $E(L)$ and $E(L')$ have the same asymptotic directions if and only if L and L' are tangent to the same inscribed deltoid $H_3(L)$.

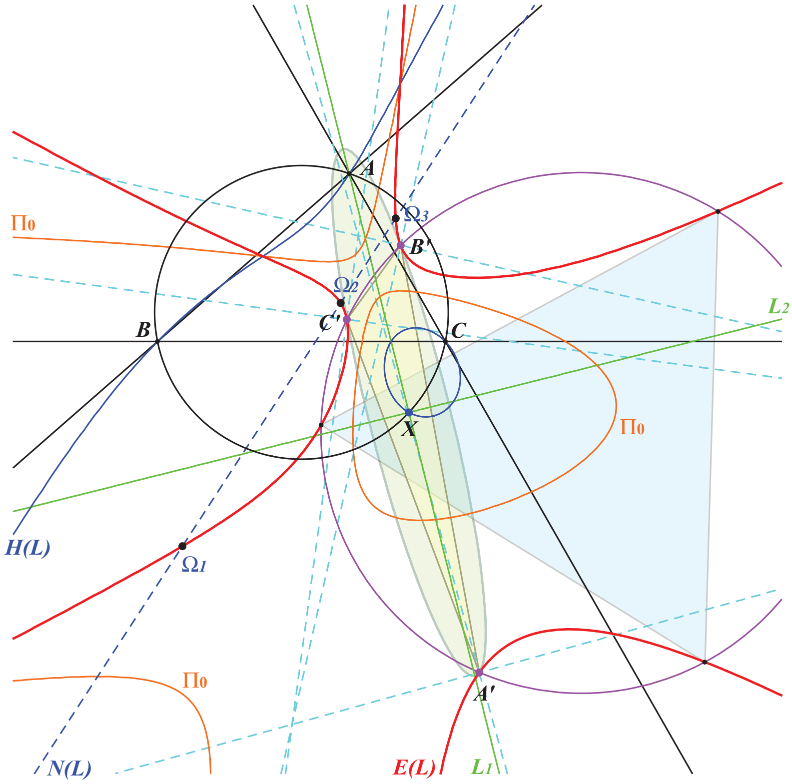

Figure 22: Cubic $E(L)$ of the Kjp type

9.3 Cubics $E(L)$ having asymptotes parallel to those of the McCay cubic

If L is the line $lx + my + nz = 0$, the asymptotes of $E(L)$ are parallel to those of the McCay cubic K003 if and only if equivalently :

- L and $N(L)$ are perpendicular,
- L is a tangent to the Steiner deltoid $\mathcal{H}_3$,
- L is the Simson line of some point on the circumcircle,
- $P = l : m : n$ lies on the Simson cubic K010. See figure 23.

9.4 Cubics $E(L)$ having asymptotes parallel to those of the Kjp cubic

If L is the line $lx + my + nz = 0$, the asymptotes of $E(L)$ are parallel to those of the Kjp cubic K024 if and only if equivalently :

- L and $N(L)$ are parallel,
- L is parallel to one sideline of triangle ABC,
- $P = l : m : n$ lies on one median of triangle ABC,
- The radial center X of $E(L)$ is one vertex of triangle ABC. See figure 24.

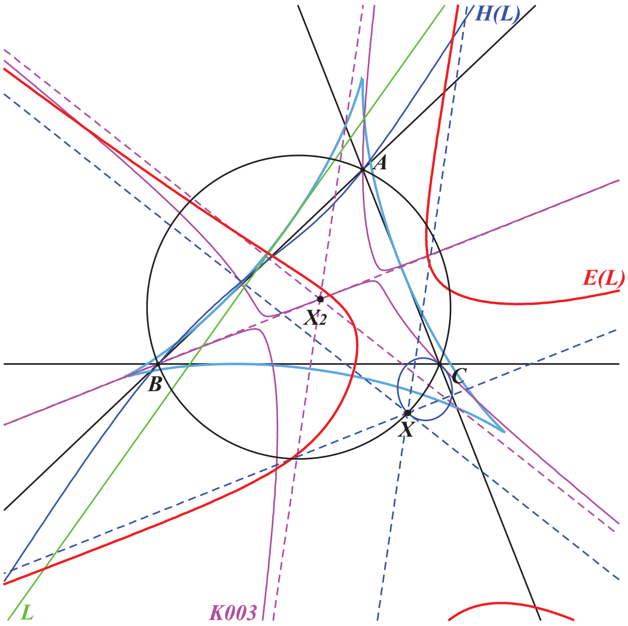

Figure 23: Cubics $E(L)$ having asymptotes parallel to those of the McCay cubic K003

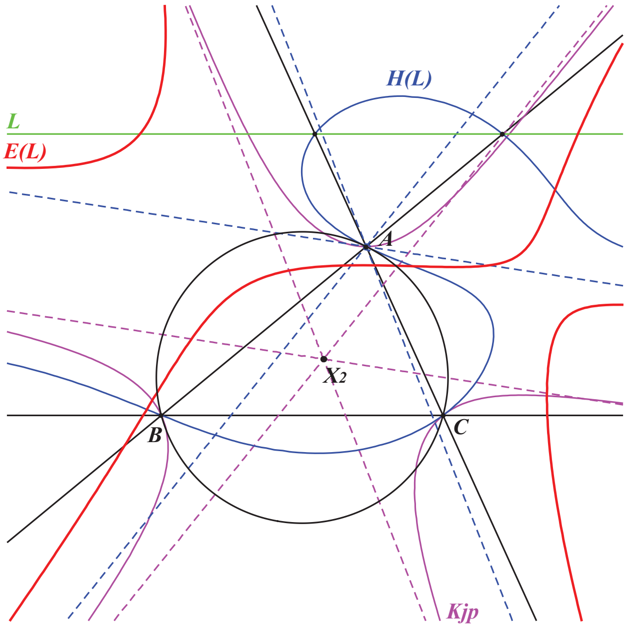

Figure 24: Cubics $E(L)$ having asymptotes parallel to those of the Kjp cubic K024

9.5 Cubics $E(L)$, $E(L')$ with same radial center having perpendicular asymptotic directions

Let $P(L)$ be the inscribed parabola tangent to the line L. If L' is the tangent to $P(L)$ perpendicular to L then the corresponding cubics $E(L)$, $E(L')$ have the same radial center X and the asymptotes of $E(L')$ are perpendicular to those of $E(L)$. These two cubics are in many ways very similar to the McCay and Kjp cubics although they do not generally intersect at three real points since they do not generally share a same group of pivots. See figure 25.

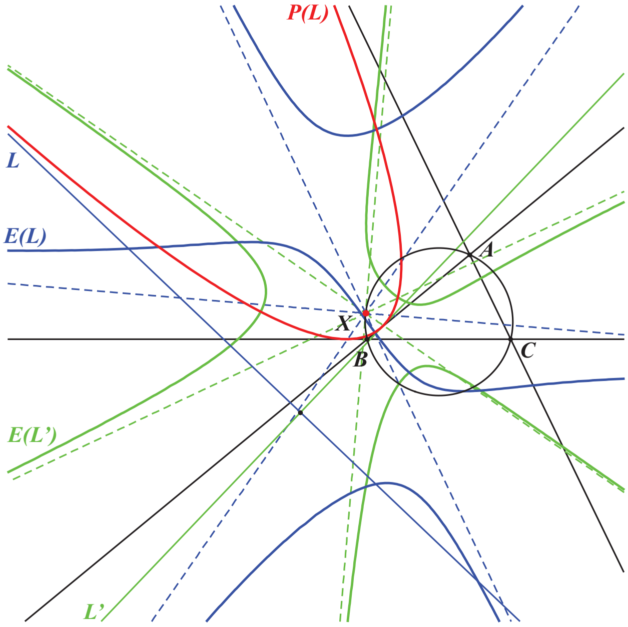

Figure 25: Cubics $E(L)$, $E(L')$ with same radial center having perpendicular asymptotic directions

9.6 Cubics $E(L)$ tangent to the line L

$E(L)$ is tangent to the line L if and only if L contains one in/excenter (which is then the point of tangency). The third point on L lies on the trilinear polar of this in/excenter.

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