Tucker cubics and Bicentric Cubics

Bernard Gibert

Created : March 26, 2002
Last update : September 30, 2015

Abstract

We explore the configurations deduced from the barycentric coordinates of a given point and define the notion of bicentric triples, bicentric conics, bicentric cubics.

1 Playing with barycentric coordinates

Let $P = (p : q : r)$ be a point distinct of the centroid G not lying on the sidelines of the reference triangle ABC (i.e. such that $pqr \neq 0$) and let ${}^tP = (1/p : 1/q : 1/r)$ be its isotomic conjugate.

1.1 Related points on the sidelines of ABC

The cevian triangle of P is denoted by $P_aP_bP_c$ with :

$$P_a = (0 : q : r), \quad P_b = (p : 0 : r), \quad P_c = (p : q : 0)$$

The cevian triangle of ${}^tP$ is denoted by ${}^tP^t_aP^t_bP_c$ with :

$${}^tP_a = (0 : r : q), \quad {}^tP_b = (r : 0 : p), \quad {}^tP_c = (q : p : 0)$$

The parallels to AB at P_b and to AC at P_c meet BC at P_{ba} and P_{ca} respectively and, similarly, the parallels to AB at ${}^tP_b$ and to AC at ${}^tP_c$ meet BC at ${}^tP_{ba}$ and ${}^tP_{ca}$ respectively. The coordinates of these four points are :

$$P_{ba} = (0 : p : r), \quad P_{ca} = (0 : q : p), \quad {}^tP_{ba} = (0 : r : p), \quad {}^tP_{ca} = (0 : p : q)$$

The corresponding points on the other sidelines of ABC are defined likewise :

$$P_{cb} = (p : 0 : q), \quad P_{ab} = (q : 0 : r), \quad {}^tP_{cb} = (q : 0 : p), \quad {}^tP_{ab} = (r : 0 : q)$$

$$P_{ac} = (r : q : 0), \quad P_{bc} = (p : r : 0), \quad {}^tP_{ac} = (q : r : 0), \quad {}^tP_{bc} = (r : p : 0)$$

We obtain a total of eighteen points on the sidelines corresponding to all the possible permutations of two non-zero coordinates taken among p, q, r.

1.2 Permuting coordinates

We can now construct all the points obtained from P by permuting its coordinates.

• When one coordinate only stays at its place, we get the three following points :

$$P_p = (p : r : q), \quad P_q = (r : q : p), \quad P_r = (q : p : r)$$

where $P_p = BP_{cb} \cap CP_{bc}, P_q = CP_{ac} \cap AP_{ca}, P_r = AP_{ba} \cap BP_{ab}$. Note that P_p is also the intersection of the line A^tP with the parallel at P to the sideline BC.

Their isotomic conjugates are obtained in the same way :

$${}^tP_p = (1/p : 1/r : 1/q) = (qr : pq : pr),$$

$$
{}^tP_q = (1/r : 1/q : 1/p) = (pq : pr : qr),
$$
$$
{}^tP_r = (1/q : 1/p : 1/r) = (pr : qr : pq)
$$

with ${}^tP_p = B^tP_{cb} \cap C^tP_{bc}$ for example.

• When none of the three coordinates stays in place, we obtain the two points :

$$
Q_1 = (q : r : p), \quad Q_2 = (r : p : q)
$$

with $Q_1 = A\,{}^tP_r \cap B\,{}^tP_p \cap C\,{}^tP_q$ and $Q_2 = A\,{}^tP_q \cap B\,{}^tP_r \cap C\,{}^tP_p$.

These points Q_1, Q_2 are called the **bicentric mates** of P and P, Q_1, Q_2 is called a **bicentric triple**. Note that the centroid of a bicentric triple is always the centroid G of ABC.

The line Q_1Q_2 is parallel to the trilinear polar of ${}^tP$ and the midpoint of Q_1Q_2 is the complement of P.

Their isotomic conjugates are

$$
{}^tQ_1 = (1/q : 1/r : 1/p) = (pr : pq : qr),
$$

$$
{}^tQ_2 = (1/r : 1/p : 1/q) = (pq : qr : pr).
$$

These two points are the perspectors of triangles ABC and $P_{ba}P_{cb}P_{ac}$, ABC and $P_{ca}P_{ab}P_{bc}$ respectively. They are sometimes called the **Brocardians** of P.

Remark : the five triangles ABC, $P_pP_qP_r$, ${}^tP_p\,{}^tP_q\,{}^tP_r$, PQ_1Q_2, ${}^tP^tQ_1^tQ_2$ have the same centroid G and the three vertices of each triangle form a bicentric triple.

Similarly, the eighteen points above also form six bicentric triples and the six corresponding triangles have the same centroid G.

• We obtain some other points by intersecting other related lines :

$$
R_p = BP_r \cap CP_q = (qr : q^2 : r^2),
$$
$$
R_q = CP_p \cap AP_r = (p^2 : pr : r^2),
$$
$$
R_r = AP_q \cap BP_p = (p^2 : q^2 : pq),
$$

and similarly their isotomic conjugates

$$
{}^tR_p = B^tP_r \cap C^tP_q = (qr : r^2 : q^2),
$$
$$
{}^tR_q = C^tP_p \cap A^tP_r = (r^2 : pr : p^2),
$$
$$
{}^tR_r = A^tP_q \cap B^tP_p = (q^2 : p^2 : pq).
$$

Note that the triangles ABC and $R_pR_qR_r$ are perspective at the point $P^2 = (p^2 : q^2 : r^2)$, the barycentric square of P.

1.3 Perspectivities

This configuration leads to a set of triply perspective triangles where the three perspectors are known points of the figure. This is summarized in the following table.

triply perspective triangles	and their three perspectors
ABC and $P_pP_qP_r$	${}^tP, {}^tQ_1$ and ${}^tQ_2$
ABC and ${}^tP_p\,{}^tP_q\,{}^tP_r$	P, Q_1 and Q_2
ABC and PQ_1Q_2	${}^tP_p, {}^tP_q$ and ${}^tP_r$
ABC and ${}^tP\,{}^tQ_1\,{}^tQ_2$	P_p, P_q and P_r
$P_pP_qP_r$ and $R_pR_qR_r$	${}^tQ_1, {}^tQ_2$ and $(p^2+qr : q^2+rp : r^2+pq)$
${}^tP_p^tP_q^tP_r$ and ${}^tR_p^tR_q^tR_r$	Q_1, Q_2 and $(p+qr/p : q+rp/q : r+pq/r)$

1.4 Special cases

1.4.1 $P = K$ (Lemoine point)

This is the Brocard configuration : ${}^tQ_1$ and ${}^tQ_2$ are the Brocard points usually denoted by Ω_1 and Ω_2 and $P_pP_qP_r$ is the first Brocard triangle $A_1B_1C_1$.

1.4.2 $P = H$ (orthocenter)

This is the configuration cited in [5] where ${}^tQ_1$ and ${}^tQ_2$ are called "cosine orthocenters" (denoted by σ_1 and σ_2 in the Tucker paper).

1.4.3 $P = I$ (incenter)

In this case ${}^tQ_1$ and ${}^tQ_2$ are sometimes called the Jerabek points.

2 Related conics

2.1 Steiner Conics

The six points $P, Q_1, Q_2, P_p, P_q, P_r$ lie on an ellipse $\mathcal{E}_1$ centered at G homothetic to the Steiner circum-ellipse with equation :

$$(qr + rp + pq)(x^2 + y^2 + z^2) = (p^2 + q^2 + r^2)(yz + zx + xy)$$

Their isotomic conjugates ${}^tP, {}^tQ_1, {}^tQ_2, {}^tP_p, {}^tP_q, {}^tP_r$ lie on another ellipse $\mathcal{E}_2$ centered at G homothetic to the Steiner circum-ellipse with equation :

$$pqr(p + q + r)(x^2 + y^2 + z^2) = (q^2r^2 + r^2p^2 + p^2q^2)(yz + zx + xy)$$

Note that these two ellipses are bicentric conics since any point $M = (x : y : z)$ on one ellipse has its two bicentric mates $M_1 = (z : x : y)$ and $M_2 = (y : z : x)$ on this same ellipse. See figure 1.

2.2 Brocard Conics

The six points $P, {}^tQ_1, {}^tQ_2, P_p, P_q, P_r$ lie on a conic $\mathcal{B}_1$ centered at $\Omega_3 = [p(2qr + p(q + r - p)) ::]$. This conic also contains G/P, the center of the circumconic with perspector P. Its equation is :

$$qrx^2 + rpy^2 + pqz^2 = p^2yz + q^2zx + r^2xy.$$

$\mathcal{B}_1$ is a circle if and only if $P = K$ (Lemoine point) and this circle is the Brocard circle with diameter OK and center X_{182}.

Similarly, the isotomic conjugates ${}^tP, Q_1, Q_2, {}^tP_p, {}^tP_q, {}^tP_r$ of the six points above also lie on a conic $\mathcal{B}_2$ centered at $\Omega_4 = [q^2r^2(qr - p(q + r + 2p)) ::]$.
Its equation is :

$$pqr(px^2 + qy^2 + rz^2) = q^2r^2yz + r^2p^2zx + p^2q^2xy.$$

This conic is the Brocard circle if and only if $P = X_{76}$.
These two conics are also bicentric conics. See figure 2.

These two conics $\mathcal{B}_1$ and $\mathcal{B}_2$ generate a pencil that contains a circle if and only if P lies either on :

1. the line at infinity in which case $\mathcal{B}_1$ decomposes into the line at infinity and a line,
2. the Steiner ellipse in which case $\mathcal{B}_2$ decomposes into the line at infinity and a line,
3. the pivotal cubic K659 $= p\mathcal{K}(G, K)$. In this latter case, the axes of the two conics have the same directions.

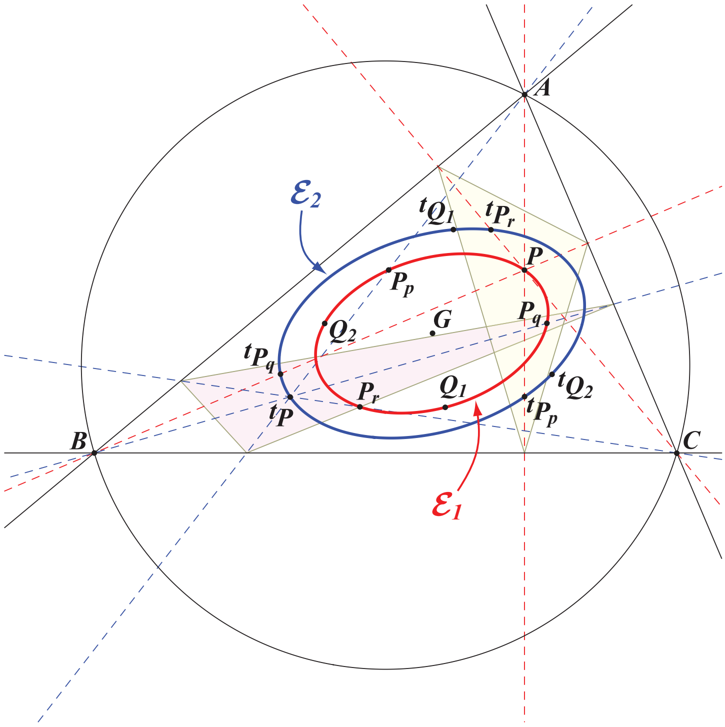

Figure 1: The Steiner Conics

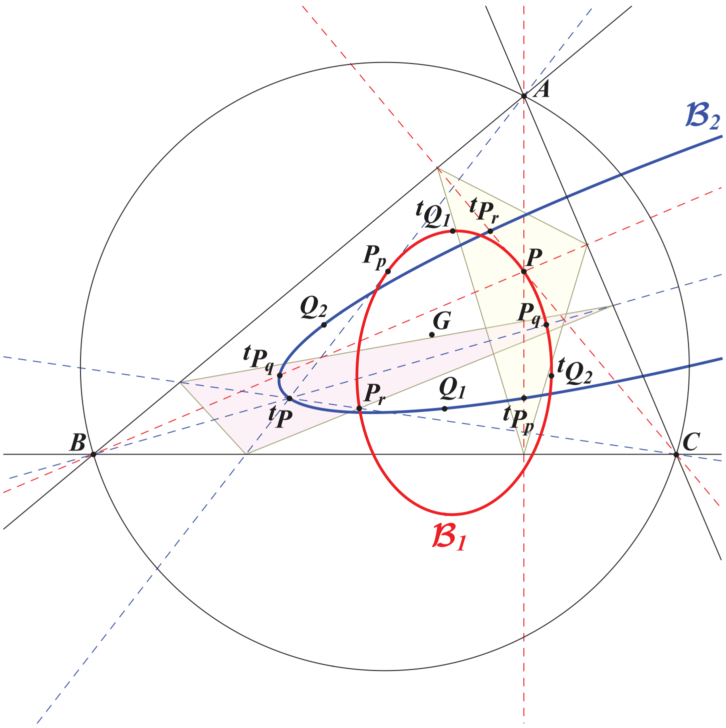

Figure 2: The Brocard Conics

2.3 Three other conics

- $P_a, P_b, P_c, {}^tP_a, {}^tP_b, {}^tP_c$ lie on the bicevian conic $\mathcal{C}_1 = \mathcal{C}(P, {}^tP)$ with equation :

$$pqr(x^2+y^2+z^2) = \sum_{\text{cyclic}} p(q^2+r^2)yz$$

$\mathcal{C}_1$ is a circle if and only if the cyclocevian conjugate of P coincide with the isotomic conjugate ${}^tP$ of P. There are six such points which are the cevian O-points of Table 39 in [1].

- $P_{ab}, P_{ba}, P_{bc}, P_{cb}, P_{ca}, P_{ac}$ lie on the bicevian conic $\mathcal{C}_2 = \mathcal{C}({}^tQ_1, {}^tQ_2)$ with equation :

$$\sum_{\text{cyclic}} qrx^2 = \sum_{\text{cyclic}} (p^2 + qr)yz$$

which is a circle if and only if $P = X_{194}$.

- Similarly ${}^tP_{ab}, {}^tP_{ba}, {}^tP_{bc}, {}^tP_{cb}, {}^tP_{ca}, {}^tP_{ac}$ lie on the bicevian conic $\mathcal{C}_3 = \mathcal{C}(Q_1, Q_2)$ with equation :

$$pqr(px^2 + qy^2 + rz^2) = \sum_{\text{cyclic}} qr(p^2 + qr)yz$$

which is a circle if and only if P is the isotomic conjugate X_{2998} of X_{194}.

Note that these three conics are bicentric conics again. See figure 3.

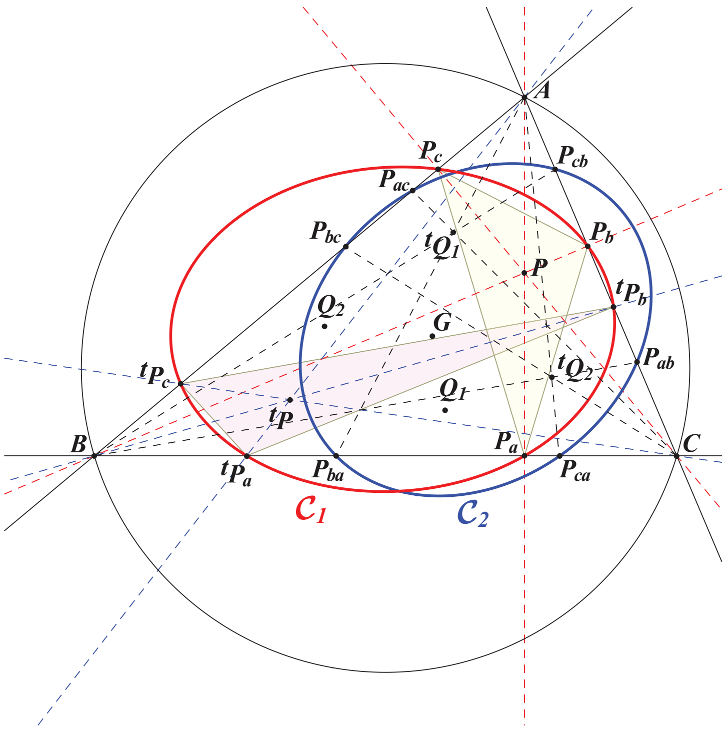

Figure 3: The Conics $\mathcal{C}_1$ and $\mathcal{C}_2$

3 Tucker cubics

3.1 A locus

• Let λ be a real number and let us denote by $\mathcal{S}$ the area of triangle ABC.

An easy computation shows that the locus of point $M = (x : y : z)$ such that the area of the cevian triangle of M is constant and equal to $\lambda\mathcal{S}$ is an isotomic non-pivotal circum-cubic $\mathcal{T}(\lambda)$ with equation :

$$\sum_{\text{cyclic}} x(y^2 + z^2) + 2\left(1 - \frac{1}{\lambda}\right) xyz = 0$$

or equivalently :

$$\lambda \sum_{\text{cyclic}} x(y^2+z^2) + 2(\lambda-1)xyz = 0,$$

or again :

$$(x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) = 1 + \frac{2}{\lambda}.$$

Its root is the centroid G.

The equations above show that all these cubics are in a pencil of circum-cubics and also that each cubic is a bicentric cubic since any point $M = (x : y : z)$ on the cubic has obviously its two bicentric mates $M_1 = (z : x : y)$ and $M_2 = (y : z : x)$ on the cubic.

• Consider the three conics γ_A, γ_B, γ_C with equations :

$$\gamma_A : \lambda(x+y)(x+z) - x^2 = 0,$$

$$\gamma_B : \lambda(y+z)(y+x) - y^2 = 0,$$

$$\gamma_C : \lambda(z+x)(z+y) - z^2 = 0.$$

γ_A passes through B, C and is centered on the median AG at the image of A in the homothety with center G, ratio $\dfrac{-2(\lambda-1)}{\lambda-4}$. γ_A is tangent at B, C to the sidelines of the antimedial triangle.

γ_A, γ_B, γ_C are parabolas if and only if $\lambda = 4$.

$\mathcal{T}(\lambda)$ is the jacobian of these three conics i.e. the locus of M such that the polar lines of M in the three conics concur at N which also lies on the cubic.

• The same locus is easily defined from a given point $P = (p : q : r)$ as the locus of point $M = (x : y : z)$ such that the (algebraic) areas of the cevian triangles of M and P are equal.

The equation given above is rewritten under the form :

$$pqr \sum_{\text{cyclic}} x(y^2+z^2) = xyz \sum_{\text{cyclic}} p(q^2+r^2)$$

$$\iff pqr \sum_{\text{cyclic}} x^2(y+z) = xyz \sum_{\text{cyclic}} p^2(q+r),$$

the cubic being then denoted by $\mathcal{T}(P)$ and called the P-Tucker cubic since it generalizes a cubic found by R. Tucker in [5].

In this case, the Tucker cubic $\mathcal{T}(P)$ is clearly the non-pivotal isotomic cubic $n\mathcal{K}(G, G, P)$.

Remark 1 : since two isotomic conjugates P and ${}^tP$ have equal area cevian triangles, we have $\mathcal{T}(P) = \mathcal{T}({}^tP)$.

Remark 2 : the third point of $\mathcal{T}(P)$ on the line through P and ${}^tP$ is Q and the tangents to the cubic at P and ${}^tP$ pass through ${}^tQ$. The coordinates are :

$$Q = \frac{p^2-qr}{p(q-r)} : \cdots : \cdots ; \quad {}^tQ = \frac{p(q-r)}{p^2-qr} : \cdots : \cdots .$$

Remark 3 : the locus of point $M = (x : y : z)$ such that the (algebraic) areas of the cevian triangles of M and P are opposite is also a cubic $\mathcal{T}'(P)$ with equation :

$$pqr \sum_{\text{cyclic}} x(y^2+z^2) = xyz \left(\sum_{\text{cyclic}} p(q^2+r^2) - 2 \prod_{\text{cyclic}} (q+r) \right).$$

This cubic is also a non-pivotal isotomic cubic with pole and root G.

3.2 Immediate properties

3.2.1 Points on the curve

It is obvious that $\mathcal{T}(P)$ passes through the eighteen following points :
– the vertices A, B, C already mentionned.
– the points at infinity of the sidelines of ABC which are inflexion points since the trilinear polar of the root G is the line at infinity.
– P and its isotomic conjugate ${}^tP$.
– the ten points obtained from P and ${}^tP$ by permuting their coordinates namely Q_1, Q_2, P_p, P_q, P_r and their isotomic conjugates ${}^tQ_1, {}^tQ_2, {}^tP_p, {}^tP_q, {}^tP_r$.
Recall that all these points come in triples formed by one point and its two bicentric mates.

3.2.2 Asymptotes

They are obviously inflexional and parallel to the sidelines of ABC. They form a triangle homothetic to ABC at G. The ratio of homothety is $\dfrac{-2(\lambda-1)}{\lambda+2}$.

3.2.3 Tangents

• The tangent at A, B, C are parallel to the sidelines of ABC and are the sidelines of the antimedial triangle. From this, $\mathcal{T}(P)$ is tritangent to the Steiner circum-ellipse at these points.

• Each median meets $\mathcal{T}(P)$ again at two points and the tangents at them are also parallel to the sidelines of ABC, this coming from the fact that the polar conic of the point at infinity of a sideline of ABC degenerates into an asymptote and the relative median.

3.2.4 Polar conic of the centroid G

The polar conic of the centroid G is generally an ellipse homothetic of the Steiner ellipse under the homothety with center G, ratio $\sqrt{\dfrac{4\lambda-1}{\lambda-1}}$. Its equation is :

$$\lambda \sum_{\text{cyclic}} x^2 + (3\lambda - 1) \sum_{\text{cyclic}} yz = 0$$

or

$$\lambda \sum_{\text{cyclic}} (x^2 - 2yz) + (5\lambda - 1) \sum_{\text{cyclic}} yz = 0$$

which clearly shows that this polar conic belongs to a pencil of concentric ellipses generated by the Steiner circum-ellipse and in-ellipse. It degenerates when $\lambda = 1$ (into the line at infinity counted twice) or when $\lambda = 1/4$ (into two imaginary lines through G and the infinite points of the Steiner ellipse). When $\lambda = 0$ it is the Steiner circum-ellipse and when $\lambda = 1/5$ it is the Steiner in-ellipse. With $\lambda = 1/3$, we find the imaginary diagonal conic with equation $x^2 + y^2 + z^2 = 0$.

When $\lambda \in [0; 1/4[\cup]1; +\infty[$, the polar conic is a real non-degenerate ellipse. Furthermore, if $\lambda \in]1; +\infty[$ the polar conic of G meets the cubic at six real points hence it is possible to draw six real tangents from G to the cubic. See for example the Tucker-Poncelet cubic below.

3.3 Construction of $\mathcal{T}(P)$

The parallel to AC at P intersects BC at B' and the parallel to AB at P intersects BC at C'.

For any point m on the line BC, let us denote by m' its homologue under the involution (on BC) which swaps B, B' and C, C'.

An easy way to realize the construction of m' is the following : the line AP intersects again the circle ABB' at E and the circle AEm meets BC again at m'.

The conic through $A, B, C, {}^tP, {}^tm$ (which is the isotomic conjugate of the line Pm) intersects the line Pm' at two points on the cubic $\mathcal{T}(P)$.

3.4 Prehessians of $\mathcal{T}(P)$

Every cubic $\mathcal{T}(P)$ has three prehessians and one of them is always real with a remarkably simple equation, namely :

$$(q+r)(r+p)(p+q)\ (x^3+y^3+z^3) = 3pqr\ (y+z)(z+x)(x+y).$$

It is interesting to observe that the mapping that sends any point of the plane to the center of its polar conic in this prehessian is the isotomic conjugation.

4 Some special examples of $\mathcal{T}(P)$

Recall that $\mathcal{T}(P)$ is actually the non-pivotal isotomic cubic $n\mathcal{K}(G, G, P)$.

4.1 The "original" Tucker cubic K011

This cubic is $\mathcal{T}(H)$ (or $\mathcal{T}({}^tH)$ where ${}^tH = X_{69}$) and is the one quoted in [5]. It contains X_4, X_{69}, X_{877}, X_{879}.

4.2 The Tucker-Brocard cubic K012

This is $\mathcal{T}(K)$ (or $\mathcal{T}({}^tK)$ where ${}^tK = X_{76}$) passing through the Brocard points and the vertices of the first Brocard triangle. It contains X_6, X_{76}, X_{880}, X_{882}.

4.3 The Tucker-Gergonne-Nagel cubic K013

This is $\mathcal{T}(X_7)$ or $\mathcal{T}(X_8)$ where X_7, X_8 are the Gergonne and Nagel points respectively. It contains X_7, X_8, X_{883}, X_{885}.

4.4 The Tucker-Jerabek cubic K014

This is $\mathcal{T}(I)$ (or $\mathcal{T}({}^tI)$ where ${}^tI = X_{75}$) passing through the Jerabek points. It contains X_1, X_{75}, X_{874}, X_{876}.

4.5 $\mathcal{T}(P)$ with concuring asymptotes K016

There is one and only one non-degenerate $\mathcal{T}(P)$ with concuring asymptotes (at G) obtained when $\lambda = 1$: this is the locus of point M such that the area of the cevian triangle of M is equal to $\mathcal{S}$.

4.6 The $G-$Tucker cubic K015

$\mathcal{T}(P)$ is unicursal if and only if it passes through one and only one fixed point of the isotomic conjugation. When this fixed point is one of the vertices of the antimedial triangle, the cubic decomposes into the union of the sidelines of this triangle.

Hence there is one and only one unicursal Tucker cubic and it is $\mathcal{T}(G)$ with a singularity at G : $\mathcal{T}(G)$ is a conico-pivotal isocubic with pivotal-conic the Steiner circum-ellipse i.e. for any point M on the curve, the line through M and its isotomic conjugate ${}^tM$ envelopes the Steiner circum-ellipse. The equation of $\mathcal{T}(G)$ is :

$$\sum_{\text{cyclic}} x(y^2+z^2) - 6\, xyz = 0$$

4.7 The Tucker-Poncelet cubic K327

This is $\mathcal{T}(2)$, locus of point M such that the area of the cevian triangle of M is $2\mathcal{S}$. It is an example of Tucker cubic such that six real tangents can be drawn through G to the cubic. See figure 4.

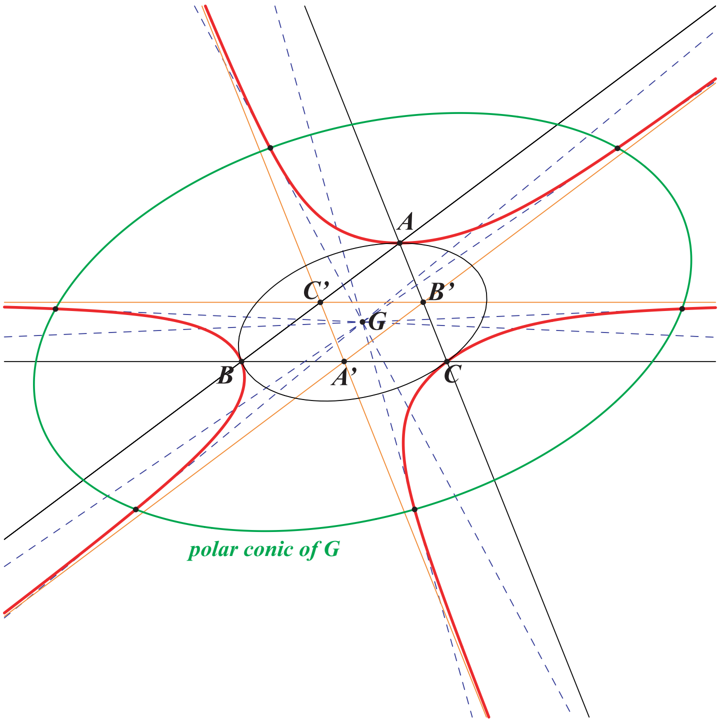

Figure 4: The Tucker-Poncelet cubic K327

5 A family of related cubics

5.1 The cubics $\mathcal{V}(P)$

Recall that the equation of the Tucker cubic $\mathcal{T}(P)$ can be written under the form

$$pqr \sum_{\text{cyclic}} x^2(y+z) - xyz \sum_{\text{cyclic}} p^2(q+r) = 0.$$

A permutation of the plus and minus signs in this equation gives

$$pqr \sum_{\text{cyclic}} x^2(y-z) + xyz \sum_{\text{cyclic}} p^2(q-r) = 0, \tag{1}$$

or equivalently

$$\prod_{\text{cyclic}} p(y-z) + \prod_{\text{cyclic}} (q-r)x = 0, \tag{2}$$

and this leads us to a new bicentric cubic $\mathcal{V}(P)$ that contains many points related with $\mathcal{T}(P)$. This cubic is a bicentric cubic again and it is in fact the pseudo-pivotal cubic $ps\mathcal{K}(G, G,^t P)$ in [2].

Indeed, $\mathcal{V}(P)$ and $\mathcal{T}(P)$ have nine identified common points namely A, B, C, ${}^tP$, P_p, P_q, P_r, ${}^tQ_1$ and ${}^tQ_2$. Note that the tangents at A, B, C are the medians of ABC.

Recall that $\mathcal{T}(P)$ is a self-isotomic non-pivotal cubic. On the other hand, the isotomic transform of $\mathcal{V}(P)$ is $\mathcal{V}({}^tP)$ with equation

$$\prod_{\text{cyclic}} p(y-z) - \prod_{\text{cyclic}} (q-r)x = 0. \tag{3}$$

which is clearly the pseudo-pivotal cubic $ps\mathcal{K}(G, G, P)$. See [2] for further properties. In particular, $\mathcal{V}(P)$ is globally unchanged under two inverse transformations which are the product in either ways of isotomic conjugation and $G-$Ceva conjugation.

Moreover, $\mathcal{V}(P)$ also contains :

- the midpoints M_a, M_b, M_c of ABC.
- G/P (the $G-$Ceva conjugate of P, the center of the circumconic $\mathcal{C}(P)$ with perspector P) and its two bicentric mates.
- $G©P$ (the $G-$cross conjugate of P) and its two bicentric mates.
- ${}^tQ_0$, the tangential of ${}^tP$, the intersection of the lines ${}^tQ_1, {}^tQ_2$ and $G/P, G©P$. See figure 5.

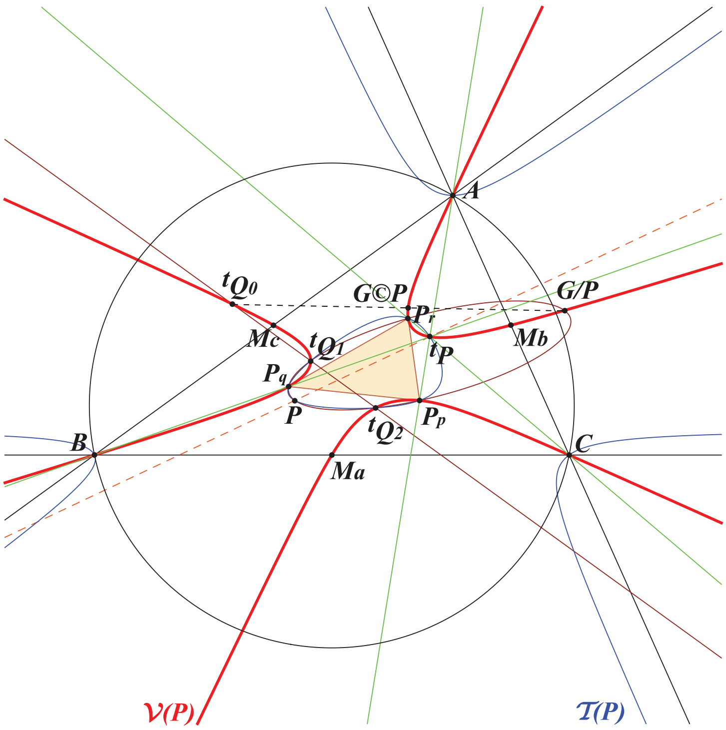

Figure 5: $\mathcal{V}(P)$ and $\mathcal{T}(P)$

- the infinite points of the pivotal cubic $\mathcal{K}_1$ with pole P and pivot the anticomplement of ${}^tP$ i.e. the $P-$Ceva conjugate of G. This easily derives from the following form of equation (1) :

$$\sum_{\text{cyclic}} (-qr + rp + pq)x(ry^2 - qz^2) = (x + y + z) \sum_{\text{cyclic}} p^2(q - r)yz.$$

$\mathcal{V}(P)$ and $\mathcal{K}_1$ meet at six other points on the circumconic $\mathcal{C}_1(P)$ with equation

$$\sum_{\text{cyclic}} p^2(q - r)yz = 0.$$

These points are A, B, C, $G©P$ and two points P_1, P_2 on the line $G, {}^tP$. Note that

– $\mathcal{C}_1(P)$ also contains P, the complement of ${}^tP$ and P^2, the barycentric square of P,

– $\mathcal{K}_1$ is a cubic with three concurring asymptotes. See figure 6. One remarkable thing to observe is

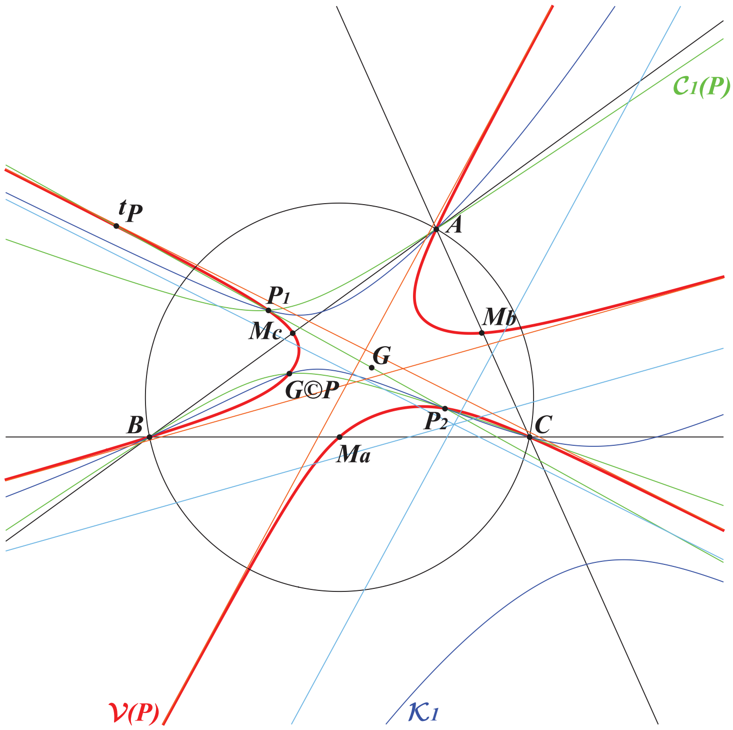

Figure 6: $\mathcal{V}(P)$ and $\mathcal{K}_1$

that the asymptotes of $\mathcal{V}(P)$ are inflexional. Indeed, the hessian cubic of $\mathcal{V}(P)$ has equation

$$\prod_{\text{cyclic}}(q-r)\left(\prod_{\text{cyclic}}p(y-z)+\prod_{\text{cyclic}}(q-r)x\right)$$
$$=4p^2q^2r^2(x+y+z)(x^2-xy+y^2-xz-yz+z^2),$$

where the left hand side represents the equation of $\mathcal{V}(P)$ itself and where $(x^2-xy+y^2-xz-yz+z^2)=0$ is the union of the two imaginary lines passing through G and the infinite points of the Steiner ellipse.

It follows that the poloconic of the line at infinity must be an ellipse with center G, homothetic to the Steiner ellipse and inscribed in the triangle formed by the asymptotes.

- the common points A, B, C, M_1, M_2, M_3 of the Steiner ellipse and the pivotal cubic $\mathcal{K}_2$ with pole P and pivot G/P. The tangents at these six points concur at G since the polar conic of G in $\mathcal{V}(P)$ is the Steiner ellipse. Note that this latter cubic has three asymptotes concurring at G.

  Furthermore, each asymptote of $\mathcal{V}(P)$ is parallel to a line passing through one vertex of ABC and one vertex of $M_1M_2M_3$. See figure 7. $\mathcal{V}(P)$ and $\mathcal{K}_2$ meet at three other points on the line $P, G/P$ namely G/P, P_3, P_4. See figure 8. This is obtained with another form of (1) :

$$\sum_{\text{cyclic}}p(-p+q+r)x(ry^2-qz^2)=(yz+zx+xy)\sum_{\text{cyclic}}qr(q-r)x.$$

- the common points of $\mathcal{C}(P)$ and the pivotal cubic $\mathcal{K}_3$ with pole P and pivot ${}^tP$. The three remaining common points are ${}^tP$ and the two points P_1, P_2 on the line $G,{}^tP$. See figure 9.

  This is obtained with another form of (1) :

$$\sum_{\text{cyclic}}qrx(ry^2-qz^2)=(pyz+qzx+rxy)\sum_{\text{cyclic}}p(q-r)x.$$

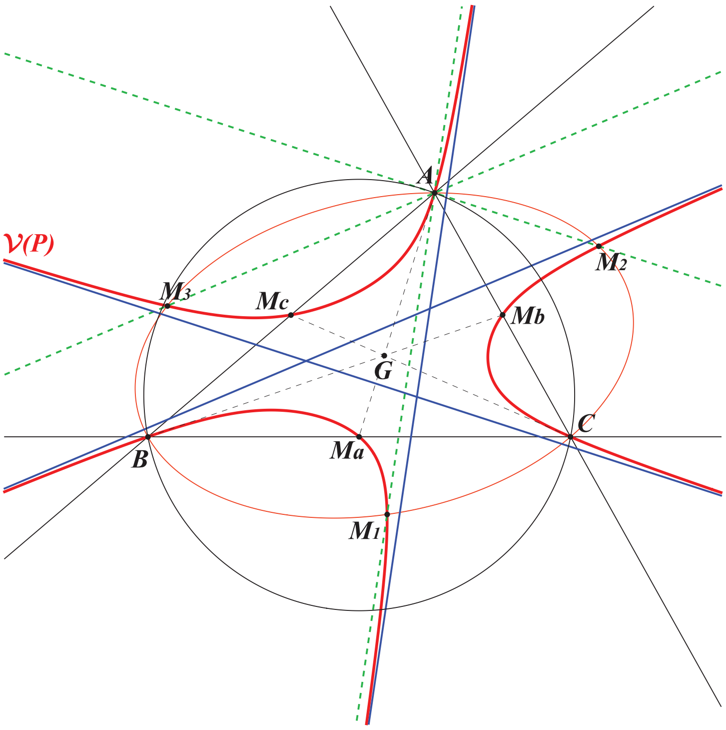

Figure 7: $\mathcal{V}(P)$ and its asymptotes

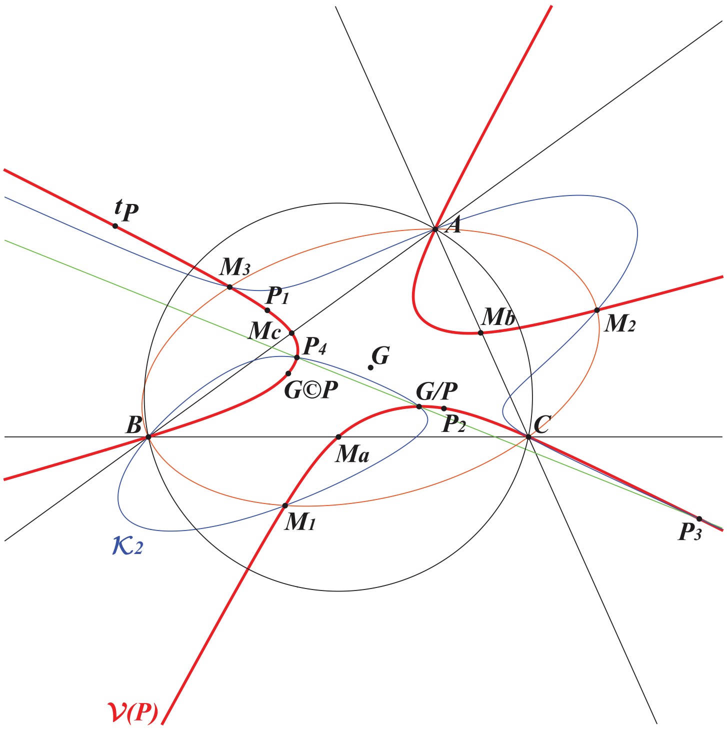

Figure 8: $\mathcal{V}(P)$ and $\mathcal{K}_2$

- the common points M_a, M_b, M_c, T_a, T_b, T_c of the inscribed Steiner ellipse and the complement $\mathcal{K}_4$ of the pivotal cubic $\mathcal{K}_2$ with pole P and pivot G/P. The three remaining common points are G/P and the two intersections P_3, P_4 of the line $P, G/P$ with the circumconic passing through G and ${}^tP$, namely the isotomic transform of the line G, P.

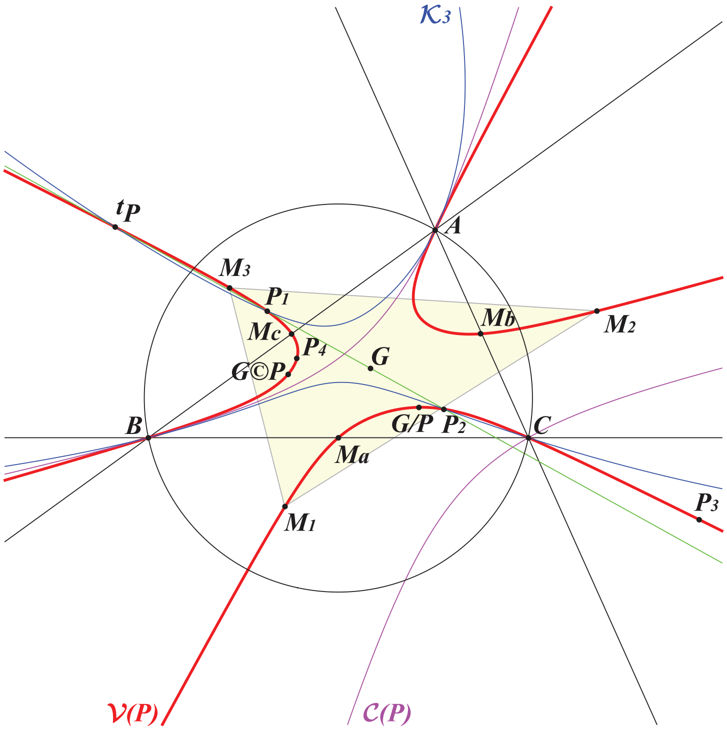

Figure 9: $\mathcal{V}(P)$ and $\mathcal{K}_3$

Note that the three points T_a, T_b, T_c are the midpoints of $M_1M_2M_3$.

This is also obtained with another form of (1) :

$$\sum_{\text{cyclic}} (2yz - x^2) \sum_{\text{cyclic}} qr(q-r)x = 4(y-z)(z-x)(x-y) - \sum_{\text{cyclic}} qr(q-r)x(-x+y+z)^2,$$

where

$$4(y-z)(z-x)(x-y) - \sum_{\text{cyclic}} qr(q-r)x(-x+y+z)^2 = 0$$

is the equation of $\mathcal{K}_4$. See figure 10.

Construction of $\mathcal{V}(P)$:

Let
– H_P be the diagonal conic through P, G/P, the vertices of the anticevian triangles of P and G/P,
– L_P be a variable line through P meeting H_P again at Q,
– C_P be the isotomic transform of L_P.
The line $Q, G/P$ meets C_P at two points of $\mathcal{V}(P)$.

Example : the Brocard-van Tienhoven cubic $\mathcal{V}(K)$ = K512

This cubic was brought to my attention by Chris van Tienhoven in a private message.

It is probably the most interesting cubic $\mathcal{V}(P)$ since it is closely related to the Brocard geometry and naturally to the Tucker-Brocard cubic K012.

Indeed, these two cubics contain the Brocard points Ω_1, Ω_2, the vertices of the first Brocard triangle, the third Brocard point X_{76}, $O = G/K$ and $X_{3224} = G©K$. See figure 11.

Furthermore, $\mathcal{V}(K)$ meets

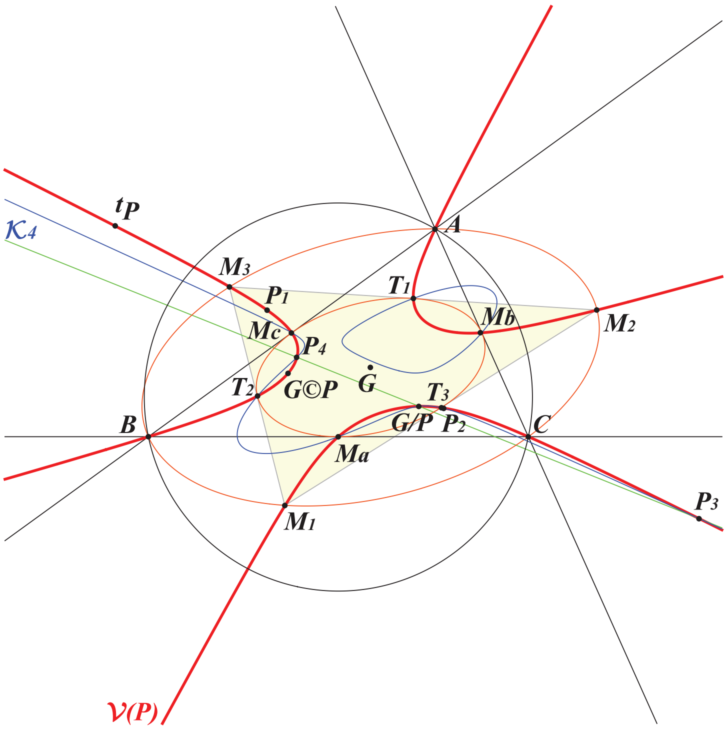

Figure 10: $\mathcal{V}(P)$ and $\mathcal{K}_4$

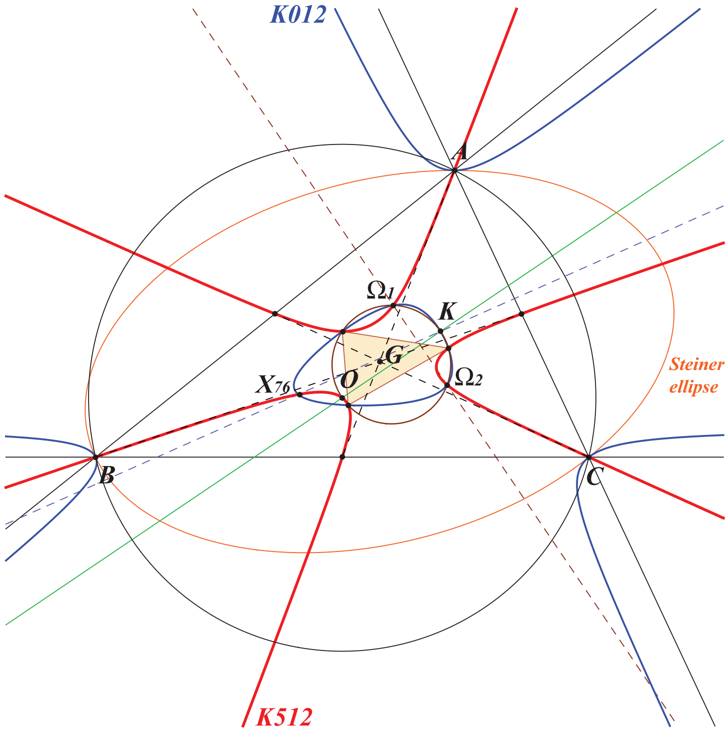

Figure 11: The Brocard-van Tienhoven cubic and K012

– the line at infinity at the same points as $p\mathcal{K}(X_6, X_{194}) =$ K410,
– the Steiner ellipse at the same points as the McCay cubic K003,
– the circumcircle of ABC at the same points as $p\mathcal{K}(X_6, X_{76}) =$ K184. See figure 12.

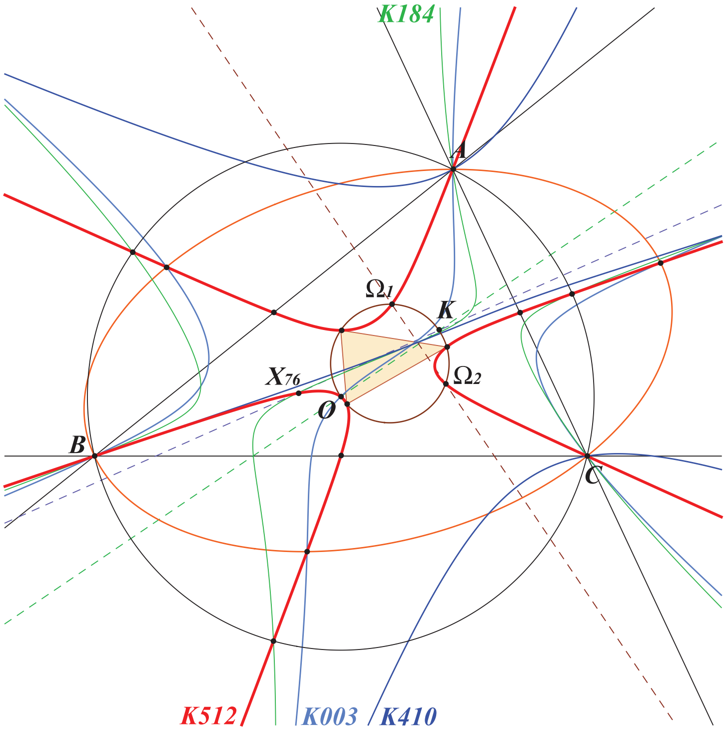

Figure 12: The Brocard-van Tienhoven cubic and K003, K184, K410

5.2 Pencils generated by $\mathcal{V}(P)$ and $\mathcal{T}(P)$

Recall that $\mathcal{V}(P)$ and $\mathcal{T}(P)$ have nine identified common points hence they generate a pencil $\mathcal{F}(P)$ of cubics passing through these nine points.

$\mathcal{F}(P)$ contains several other remarkable cubics.

- one cubic $\mathcal{V}^+(P)$ without term in xyz (a $\mathcal{K}_0$ cubic) which is at the same time a $\mathcal{K}^+$ i.e. having three concurring asymptotes, here in G. See figure 13.

  The asymptotes of $\mathcal{V}^+(P)$ are also inflexional. Indeed, the hessian cubic of $\mathcal{V}^+(P)$ has equation

  $$(x+y+z)(x^2-xy+y^2-xz-yz+z^2)=0$$

  which we have already met above.

- one cubic $\mathcal{V}'(P)$ inscribed in the antimedial triangle $G_aG_bG_c$. This also contains the anticomplement of the isotomic conjugate of the complement of P. The tangents at G_a, G_b, G_c pass through G. See figure 14.

6 General bicentric circum-cubics

It is obvious that a bicentric circum-cubic $\mathcal{B}_{u,v,w}$ must have an equation of the form :

$$u\,(x^2y+y^2z+z^2x)+v\,(xy^2+yz^2+zx^2)+4w\,xyz=0,$$

where u, v, w are any three real (not all zero) numbers.

Note that $\mathcal{B}_{u,v,w}$ meets the sidelines of ABC at three points independent of w namely :
$U=(0:-v:u)$, $V=(u:0:-v)$ and $W=(-v:u:0)$.

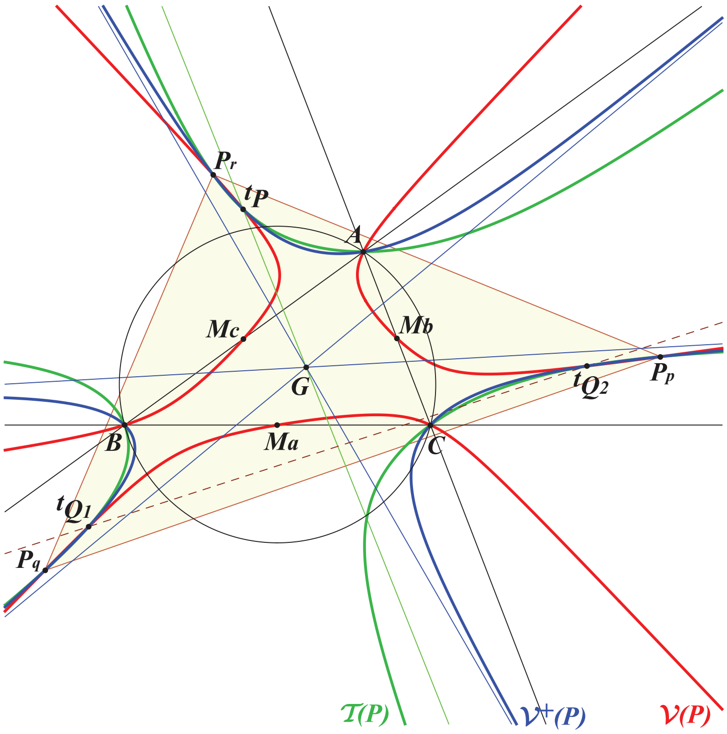

Figure 13: $\mathcal{V}^+(P)$, $\mathcal{V}(P)$ and $\mathcal{T}(P)$

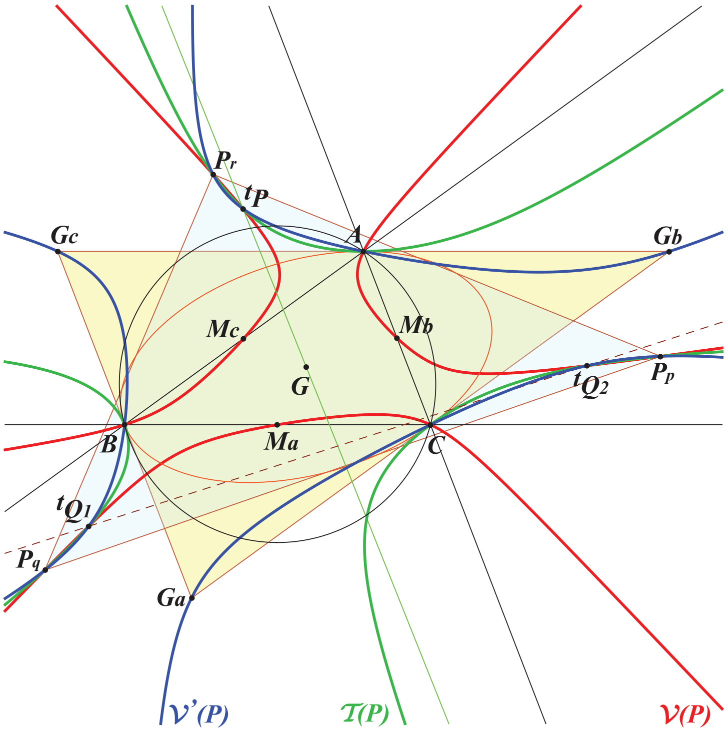

Figure 14: $\mathcal{V}'(P)$, $\mathcal{V}(P)$ and $\mathcal{T}(P)$

These points form a bicentric triple with centroid G and, moreover, they are the tangentials of the vertices of ABC i.e. the tangents at A, B, C to $\mathcal{B}_{u,v,w}$ pass through U, V, W respectively.

More generally, for any point P on $\mathcal{B}_{u,v,w}$, the triangle formed by P and its bicentric mates is inscribed in the cubic and rotates around G when P traverses the cubic.

The isotomic transform of $\mathcal{B}_{u,v,w}$ is clearly $\mathcal{B}_{v,u,w}$ and this shows that $\mathcal{B}_{u,v,w}$ is a self-isotomic cubic if and only if $u = v$. These are the Tucker cubics. This is the only case when the points U, V, W are collinear.

The points U, V, W are the vertices of a cevian triangle if and only if $u + v = 0$, and with w suitably chosen, we obtain the cubics $\mathcal{V}(P)$ seen above. In the particular case $w = 0$, we obtain $\mathcal{V}^{+}(P)$.

6.1 Polar conic of the centroid

The polar conic $\mathcal{C}_{u,v,w}$ of G with respect to $\mathcal{B}_{u,v,w}$ can be written under the form :

$$(u + v)(x + y + z)^2 + 4w(yz + zx + xy) = 0, \tag{4}$$

showing that it is generally an ellipse with center G homothetic to the Steiner ellipse. This can be real or not, proper or degenerate.

The ratio ρ of homothecy is given by $\rho^2 = \dfrac{3u + 3v + 4w}{4w}$.

Naturally $\mathcal{B}_{u,v,w}$ meets $\mathcal{C}_{u,v,w}$ at six points with tangents concurring at G. The equation (4) and the ratio ρ clearly confirm that :

- when $u + v = 0$, $\mathcal{C}_{u,v,w}$ is the Steiner ellipse ($\rho = 1$),
- when $w = 0$, $\mathcal{B}_{u,v,w}$ has three inflexional asymptotes concurring at G since $\mathcal{C}_{u,v,w}$ is the line at infinity counted twice ($\rho = \infty$),
- when $u + v + w = 0$, $\mathcal{C}_{u,v,w}$ is the Steiner in-ellipse ($\rho = 1/2$),
- when $u + v = 4w$, $\mathcal{C}_{u,v,w}$ is the Steiner ellipse of the antimedial triangle ($\rho = 2$),
- when $u + v + 2w = 0$, $\mathcal{C}_{u,v,w}$ is the diagonal conic $x^2 + y^2 + z^2 = 0$ ($\rho^2 = -1/2$).

Each condition above leads to a pencil of cubics such that the polar conic of G is the same for any cubic in the pencil.

6.2 Bicentric circum-cubics passing through a given point

Let us now impose that the cubic $\mathcal{B}_{u,v,w}$ contains a given point P. It is clear from the general equation above that the corresponding cubics now form a pencil passing through nine fixed points forming three bicentric triples. These triples are A, B, C – P, Q_1, Q_2 – ${}^tP_p$, ${}^tP_q$, ${}^tP_r$.

Each cubic of the pencil can be characterized by a single parameter and, when this parameter varies, the polar conic of G meets the corresponding cubic at six points lying on a same bicentric circum-quintic $\mathcal{Q}(P)$ that contains P, Q_1, Q_2, ${}^tP_p$, ${}^tP_q$, ${}^tP_r$, the infinite points of the Steiner ellipse, G, G_a, G_b, G_c, the anticomplement aP of P and its two bicentric mates.

This quintic has five asymptotes concurring at G. Note that the tangents at A, B, C, P, Q_1, Q_2, ${}^tP_p$, ${}^tP_q$, ${}^tP_r$ pass through G. See figure 15.

6.3 Examples

We conclude this paper with several selected examples and some additional remarks concerning each particular situation..

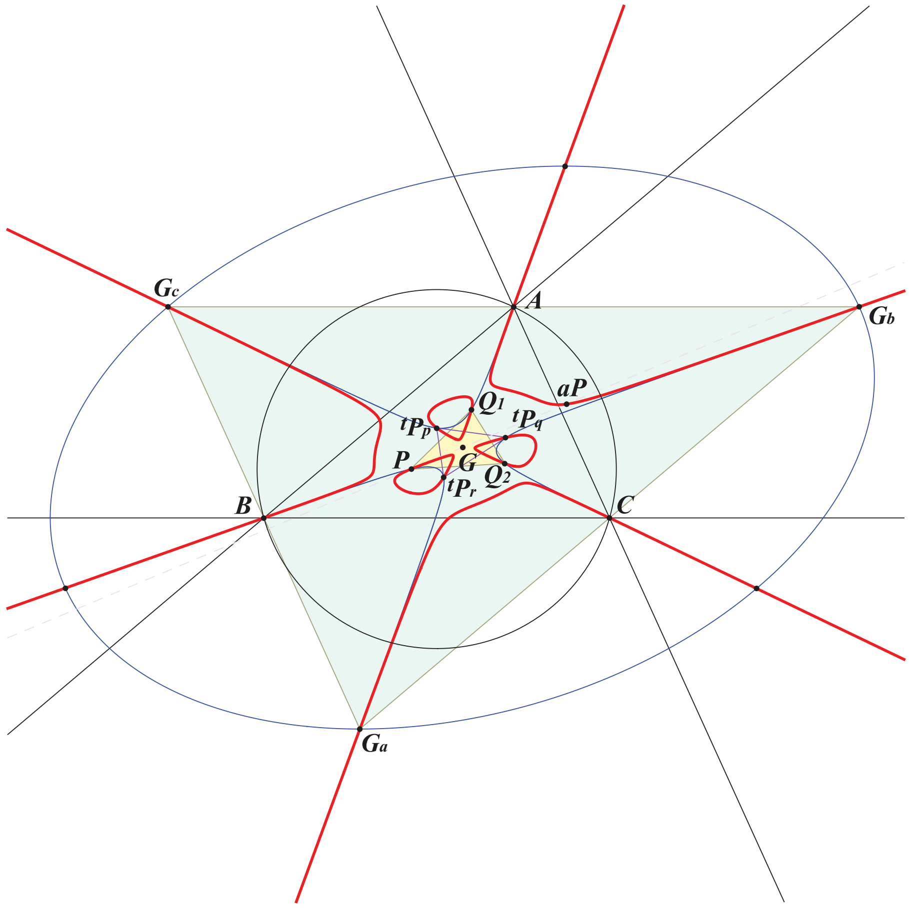

Figure 15: The bicentric circum-quintic $\mathcal{Q}(P)$

6.3.1 Example 1

When $\mathcal{C}_{u,v,w}$ is the Steiner ellipse, the cubic $\mathcal{B}_{u,v,w}$ must pass through the midpoints of ABC and thus the tangents at A, B, C must be the medians of ABC. $\mathcal{B}_{u,v,w}$ meets the Steiner ellipse at three other points A', B', C' forming a bicentric triple with centroid G.

All these cubics form a pencil and the cubic that passes through the given point $P = (p : q : r)$ has equation :

$$\sum_{\text{cyclic}} px(ry - qy)(qy + rz) = 0.$$

It also contains the $G-$Ceva conjugate of ${}^tP$ and the $G-$crossconjugate of ${}^tP$ (equivalently the isotomic conjugate of the $G-$Ceva conjugate of P) and the four corresponding bicentric mates. The tangential of P is the third point on the line passing through the two former points.

The figure 16 shows the cubic that contains X_9, X_{75}, X_{87} and naturally their six bicentric mates.

6.3.2 Example 2

When $\mathcal{C}_{u,v,w}$ is the Steiner in-ellipse, the cubic $\mathcal{B}_{u,v,w}$ passing through P meets this ellipse at six (real or not) points with their tangents passing through G. These six points form two other bicentric triples.

The figure 17 shows the cubic that contains X_{76}, the Brocard points, the vertices of the first Brocard triangle.

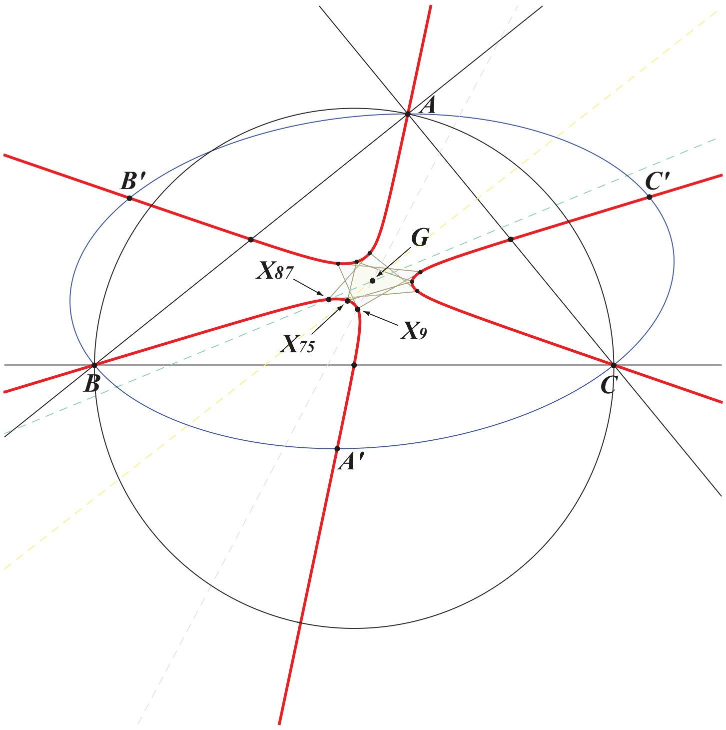

Figure 16: $\mathcal{B}_{u,v,w}$ with polar conic the Steiner ellipse

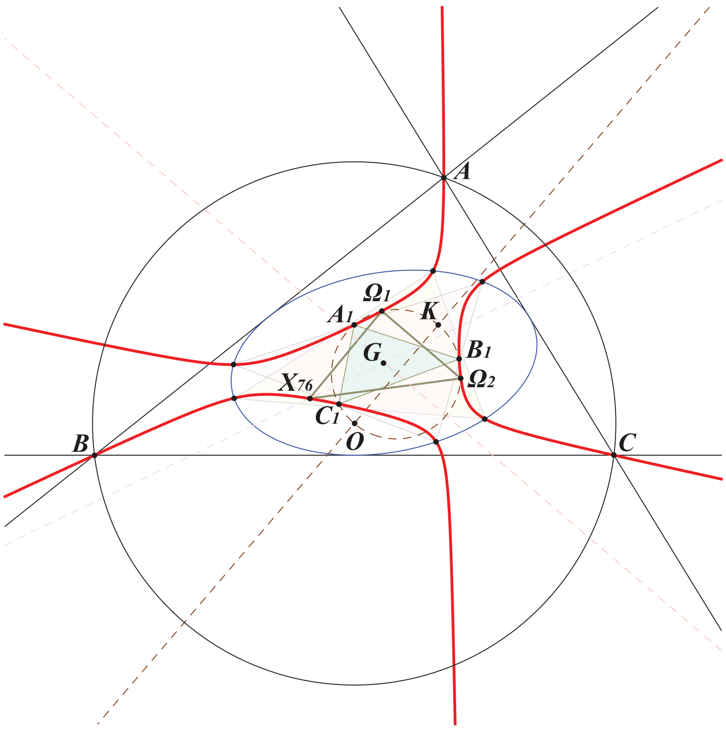

Figure 17: $\mathcal{B}_{u,v,w}$ with polar conic the Steiner in-ellipse

6.3.3 Example 3

When $\mathcal{C}_{u,v,w}$ is the union of the two imaginary lines passing through G and the infinite points of the Steiner ellipse, the cubic $\mathcal{B}_{u,v,w}$ is an acnodal cubic with G as isolated node on the curve.

The figure 18 shows the cubic that contains X_{76}, the Brocard points, the vertices of the first Brocard triangle.

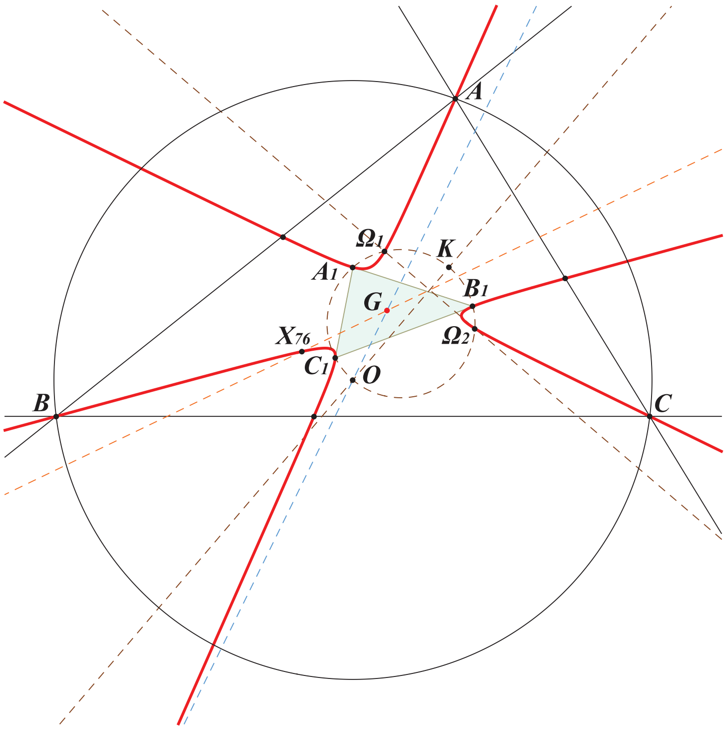

Figure 18: $\mathcal{B}_{u,v,w}$ with a degenerated polar conic

6.3.4 Example 4

When $\mathcal{C}_{u,v,w}$ is the Steiner ellipse of the antimedial triangle $G_aG_bG_c$, the cubic $\mathcal{B}_{u,v,w}$ passing through P also contains G_a, G_b, G_c and naturally Q_1, Q_2, ${}^tP_p$, ${}^tP_q$, ${}^tP_r$, these nine points forming three bicentric triples. Figure 19 shows the cubic that contains X_{69}.

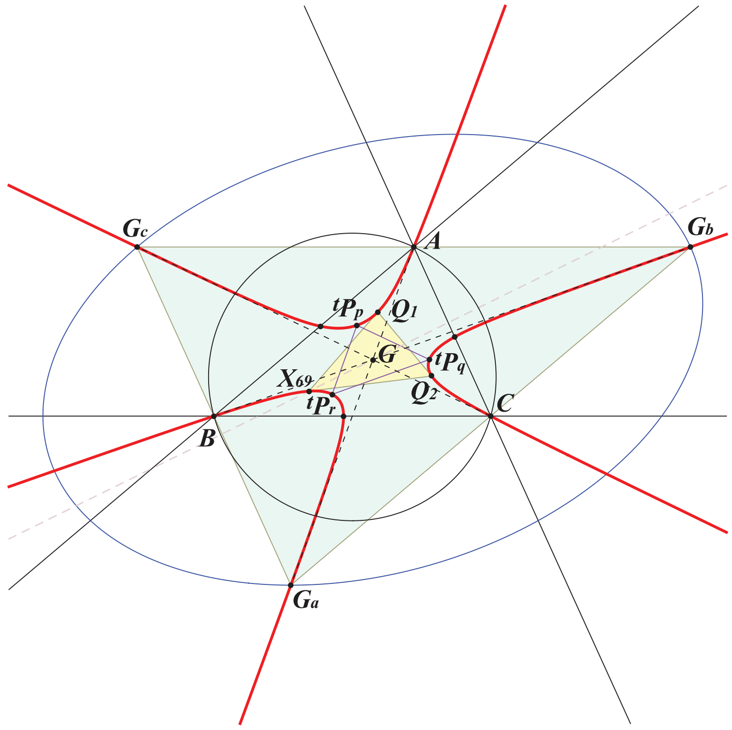

Figure 19: $\mathcal{B}_{u,v,w}$ with polar conic the Steiner ellipse of the antimedial triangle

7 Generalized Tucker cubics

The Tucker cubics $\mathcal{T}(P) = n\mathcal{K}(G, G, P)$ we met above are deeply related to the centroid G of triangle ABC through isotomic conjugation and therefore to the line at infinity, the trilinear polar of G that contains the three real inflexion points of $\mathcal{T}(P)$. The corresponding cubics $\mathcal{V}(P)$ have the same relationship with G since they are pseudo-pivotal cubics with pseudo-pole and pseudo-pivot G. See [2] for more details. Recall that both cubics are bicentric cubics.

In this section, G is replaced by any other point Z not lying on one sideline of ABC and the line at infinity by the trilinear polar $\mathcal{L}_Z$ of Z meeting the sidelines of ABC at U, V, W. The isotomic conjugation is now the isoconjugation with fixed points Z and its three harmonic associates. The isoconjugate of any point M is still denoted by ${}^tM$.

All the points defined at the beginning are here defined similarly since the parallels to the sidelines of ABC are replaced by lines intersecting on $\mathcal{L}_Z$. More precisely, the parallels at M to BC, CA, AB are now the lines UM, VM, WM respectively.

The $Z-$Tucker cubic $\mathcal{T}(Z, P) = n\mathcal{K}(Z^2, Z, P)$ is now :

$$\left(\frac{x}{u} + \frac{y}{v} + \frac{z}{w}\right)\left(\frac{u}{x} + \frac{v}{y} + \frac{w}{z}\right) = \left(\frac{p}{u} + \frac{q}{v} + \frac{r}{w}\right)\left(\frac{u}{p} + \frac{v}{q} + \frac{w}{r}\right)$$

$$\Longleftrightarrow\ pqr \sum_{\text{cyclic}} vwx^2(wy + vz) - xyz \sum_{\text{cyclic}} p^2vw(wq + vr) = 0,$$

and the corresponding cubic $\mathcal{V}(Z, P) = ps\mathcal{K}(Z^2, Z,{}^t P)$ is :

$$pqr \sum_{\text{cyclic}} vwx^2(wy - vz) + xyz \sum_{\text{cyclic}} p^2vw(wq - vr) = 0$$

$$\Longleftrightarrow\ pqr \prod_{\text{cyclic}} (wy - vz) + xyz \prod_{\text{cyclic}} (wq - vr) = 0,$$

where Z^2 denotes the barycentric square of Z.
Note that these two cubics are not bicentric cubics anymore.

References

[1] Gibert B. *Cubics in the Triangle Plane*, available at
http://pagesperso-orange.fr/bernard.gibert/

[2] Gibert B. *Pseudo-Pivotal Cubics and Poristic Triangles*, available at
http://pagesperso-orange.fr/bernard.gibert/

[3] Kimberling C., *Triangle Centers and Central Triangles*, Congressus Numerantium, 129 (1998) 1–295.

[4] Kimberling C., *Encyclopedia of Triangle Centers*, 2000-2009
http://www2.evansville.edu/ck6/encyclopedia
http://faculty.evansville.edu/ck6/encyclopedia/ETC.html

[5] Tucker R., *The "cosine" orthocentres of a triangle and a cubic through them.* [J] Mess. (2) XVII. 97-103, (1887).