

Cubics Related with Radical Axes

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Last update : April 24, 2008

Abstract

This paper is a complement of [5]. It studies several families of cubics related with radical axes and pedal circles.

1 The net $\mathcal{K}_1(P)$

1.1 Generalities

In the plane of the reference triangle ABC , let $P = p : q : r$ be a fixed point and M a variable point with pedal circle $\mathcal{C}(M)$. Let us denote by $\mathcal{A}(M)$ the radical axis of $\mathcal{C}(M)$ and $(\mathcal{O})$, the circumcircle of ABC . Recall that two isogonal conjugate points M and M^* share the same pedal circle and that when one these points, say M , lies on $(\mathcal{O})$ the pedal circle degenerates into the line at infinity $\mathcal{L}^\infty$ and the Simson line $\mathcal{S}(M)$ of M .

We seek the locus of M for which $\mathcal{A}(M)$ passes through the fixed point P .

Theorem 1. *The radical axis of the circumcircle $(\mathcal{O})$ of ABC and the pedal circle of a point M passes through a fixed point P if and only if M lies on an isogonal non-pivotal cubic $\mathcal{K}_1(P)$.*

This cubic $\mathcal{K}_1(P)$ has barycentric equation :

$$\sum p x (S_A y + b^2 z)(S_A z + c^2 y) = 0 \quad (1)$$

where $2S_A = b^2 + c^2 - a^2$, S_B and S_C similarly.

Equivalent definitions

- Following [3], §1.5.5, $\mathcal{K}_1(P)$ is also the locus of M such that $\mathcal{C}(M)$ is orthogonal to the fixed circle Γ_P with center P , orthogonal to $(\mathcal{O})$.

Indeed, Γ_P being orthogonal to $(\mathcal{O})$ and $\mathcal{C}(M)$ must have its center on the radical axis of these two latter circles i.e. P must lie on $\mathcal{A}(M)$.

In [1], Bouvaist calls P the *orthopole* of the cubic and Γ_P its *orthopolar circle* by analogy with the decomposed cubic union of a line and the circum-conic which is its isogonal transform. This is a consequence of Lemoyne's theorem on the orthopole.

- Following [3], §1.5.3, $\mathcal{K}_1(P)$ is also the locus of M such that M and its isogonal conjugate M^* are conjugated with respect to another fixed circle Γ'_P whose center is the reflection P' of H in P . Γ'_P is orthogonal to the circle with center O and radius $\mathcal{R}\sqrt{1 + 4 \cos A \cos B \cos C}$ where $\mathcal{R}$ is the radius of $(\mathcal{O})$. It belongs to the pencil of circles containing Γ_P and the degenerate circle which is the union of $\mathcal{L}^\infty$ and the Newton line of the complete quadrilateral formed by the sidelines of ABC and $\mathcal{L}(P)$. See below.

Properties of $\mathcal{K}_1(P)$

The equation (1) clearly shows that these cubics $\mathcal{K}_1(P)$ form a net of circum-cubics which can be generated by three decomposed cubics. One of them is $\mathcal{K}(A)$, the union of the sideline BC and two other lines through A with equations $S_A y + b^2 z = 0$ and $S_A z + c^2 y = 0$. These latter two lines are isogonal transforms of one another. They are the perpendiculars at A to the sidelines AB and AC .

$\mathcal{K}_1(P)$ meets the sidelines of ABC at three collinear points U, V, W lying on a line $\mathcal{L}(P)$ whose trilinear pole R is called the root of the cubic. This root is the barycentric quotient of P and H , the orthocenter of ABC . It has coordinates $R = S_A p : S_B q : S_C r$. Equivalently, R is

the barycentric product of P and X_{69} , the isotomic conjugate of H . Remark that U is also the intersection of the sideline BC with the polar line of A in Γ_P i.e. the perpendicular to PA at its second intersection with $(\mathcal{O})$. V and W are defined in the same way.

The Newton line of the complete quadrilateral formed by the sidelines of ABC and $\mathcal{L}(P)$ is the orthic line of the cubic. It is the locus of points whose polar conic in the cubic is a rectangular hyperbola. It is also the orthic axis of the triangle formed by the asymptotes.

$\mathcal{K}_1(P)$ meets $(\mathcal{O})$ at A, B, C and three other points Q_1, Q_2, Q_3 , one of these points being always real. The pedal circle of Q_i is the union of $\mathcal{L}^\infty$ and the Simson line $\mathcal{S}_i$ of Q_i which is at the same time the radical axis of $(\mathcal{O})$ and the (degenerate) pedal circle of Q_i . Hence, the number of real points Q_i is the number of tangents that can be drawn from P to the Steiner deltoid $\mathcal{H}_3$. It follows that these points are all real if and only if P lies inside $\mathcal{H}_3$. This is in particular the case when P is the nine point center X_5 which will be examined below.

Obviously, the number of real asymptotes is the number of these real points Q_i .

Construction of $\mathcal{K}_1(P)$

We give a step by step construction of the cubic adapted from [3], §1.5.4.

1. Given P , construct the line $\mathcal{L}(P)$ through U, V, W as seen above.
2. Construct the radical center X of $(\mathcal{O})$ and two in/excircles, say $(\mathcal{I})$ and $(\mathcal{I}_a)$. Note that the radical axis of $(\mathcal{O})$ and the pedal circle of any point on the line AI passes through X .
3. Draw the line PX and its perpendicular at O meeting AI at the center of the pedal circle of two isogonal conjugate points Q, Q^* on AI and on $\mathcal{K}_1(P)$. This pedal circle must pass through the common points of $(\mathcal{O})$ and PX . If these points are not real, choose another pair of in/excircles.
4. The cevian lines of Q in ABC meet $\mathcal{L}(P)$ at Q_a, Q_b, Q_c . Take m on $\mathcal{L}(P)$ and construct its homologue m' in the involution swapping U and Q_a, V and Q_b, W and Q_c .
5. The line Qm' meets the circumconic through Q^* and m^* at two points M, N on $\mathcal{K}_1(P)$. In the same way, the line Qm meets the circumconic through Q^* and m'^* at their isogonal conjugates M^*, N^* .

1.2 Singular cubics

The discriminant of $\mathcal{K}_1(P)$ is a polynomial of global degree 12 in p, q, r . It factorizes into four factors of degree 1 and the square of a factor of degree 4. When this discriminant vanishes, the cubic has at least one singular point. If one can find two distinct singular points on the cubic, it must decompose. It follows that $\mathcal{K}_1(P)$ is singular if and only if P lies on four lines L_o, L_a, L_b, L_c or on a trinodal quartic $\mathcal{Q}$. See figure 1.

These lines turn out to be the radical axes of $(\mathcal{O})$ and the in/excircles. These are also the trilinear polar L_o of X_{279} and the trilinear polars L_a, L_b, L_c of the extraversions of X_{279} .

One can see that P lies on one of these lines if and only if $\mathcal{K}_1(P)$ passes through the corresponding in/excenter of ABC . In such case, $\mathcal{K}_1(P)$ becomes a $c\mathcal{K}$ i.e. a conico-pivotal (unicursal) cubic with node the mentioned in/excenter. See [3], chapter 8.

L_o contains $X_{513}, X_{676}, X_{2488}, X_{2494}, X_{2495}, X_{2496}, X_{2497}, X_{2498}, X_{2499}, X_{2520}, X_{2976}$ and each of these points gives a nodal $\mathcal{K}_1(P)$ with node X_1 . These cubics form a pencil containing one circular cubic $\mathcal{K}_1(X_{513})$. See below.

The quartic $\mathcal{Q}$ is the locus of P such that the cubic contains a fourth point on the line $\mathcal{L}(P)$. In other words, $\mathcal{L}(P)$ decomposes into $\mathcal{L}(P)$ and the circum-conic which is its isogonal conjugate. $\mathcal{Q}$ is the isogonal transform of a conic with center O . $\mathcal{Q}$ is bitangent to each of the four lines above. However, the points of tangency are not always all real.

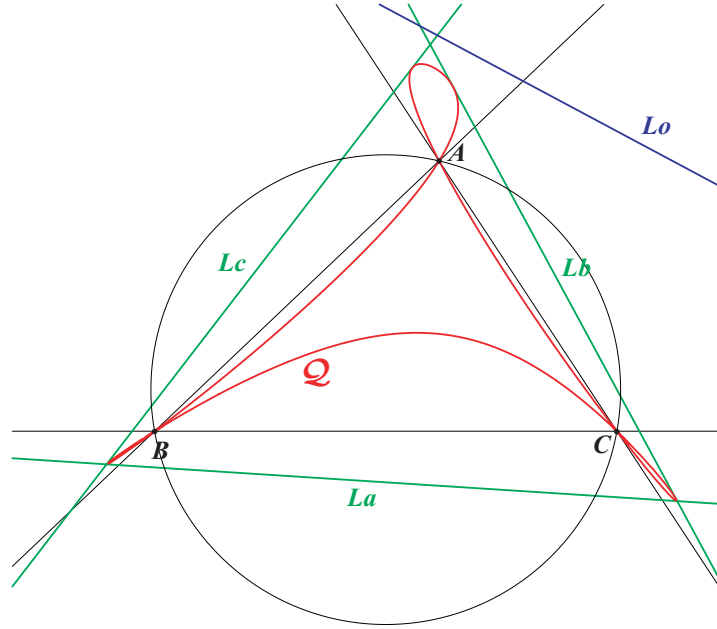


Figure 1: The quartic $\mathcal{Q}$ and the four lines L_o, L_a, L_b, L_c

1.3 Circular focal cubics

$\mathcal{K}_1(P)$ is a circular cubic if and only if P lies on $\mathcal{L}^\infty$ hence the root R lies on the trilinear polar of X_{69} , the isotomic conjugate of H . In this case, the singular focus F is the reflection in O of the isogonal conjugate P^* of P . Since F lies on $\mathcal{K}_1(P)$ (and on $(\mathcal{O})$), the cubic is a focal cubic.

These focal cubics form a pencil of cubics passing through the foci of the inscribed conic with center O . Their axes (or orthic lines) are the parallels at O to the Simson lines of P^* . It follows that the real asymptote is the parallel at P^* (the antipode of F on $(\mathcal{O})$) to the Simson lines of P^* . Each cubic can be seen as the locus of foci of inscribed conics centered on a line passing through O . When this line is the Brocard axis the cubic is **K019** (see below) and when it is the Euler line the cubic is **K187**, the Euler isogonal focal cubic.

When P is the infinite point of one of the lines L_o, L_a, L_b, L_c the cubic is a strophoid. Thus, there are exactly four strophoids in the net. If P is X_{513} , the infinite point of L_o and also the infinite point of the antiorthic axis, the strophoid is $c\mathcal{K}(\#X_1, X_{905})$. This strophoid is the locus of foci of inscribed conic with center on the line through the incenter I and the circumcenter O of ABC . See figure 2 and [3], §4.3.2 for further details and a construction.

1.4 $n\mathcal{K}_0$ cubics

$\mathcal{K}_1(P)$ is a $n\mathcal{K}_0$ (without term in xyz) if and only if P lies on a line perpendicular to the Brocard axis. Thus, these $n\mathcal{K}_0$ form a pencil of cubics.

In particular, there is one and only one circular (focal) $n\mathcal{K}_0$ obtained when $P = X_{512}$. This is **K019** = $n\mathcal{K}_0(X_6, X_{647})$, the third Brocard cubic. Recall that this cubic is the locus of foci of inscribed conics in ABC whose center is a point on the Brocard axis OK . Thus, **K019** contains in particular the Brocard points and the four foci of the “K-ellipse”, the inconic with center K .

1.5 Equilateral cubic

$\mathcal{K}_1(P)$ is an equilateral cubic (or a $\mathcal{K}_{60}$) if and only if P is X_5 , the nine point center. The corresponding root is X_{343} . This cubic meets the line at infinity $\mathcal{L}^\infty$ and $(\mathcal{O})$ at the same nine points as the Kjp cubic **K024** = $n\mathcal{K}_0^+(X_6, X_6)$. The equilateral triangle formed by the asymptotes has center G . See figure 3.

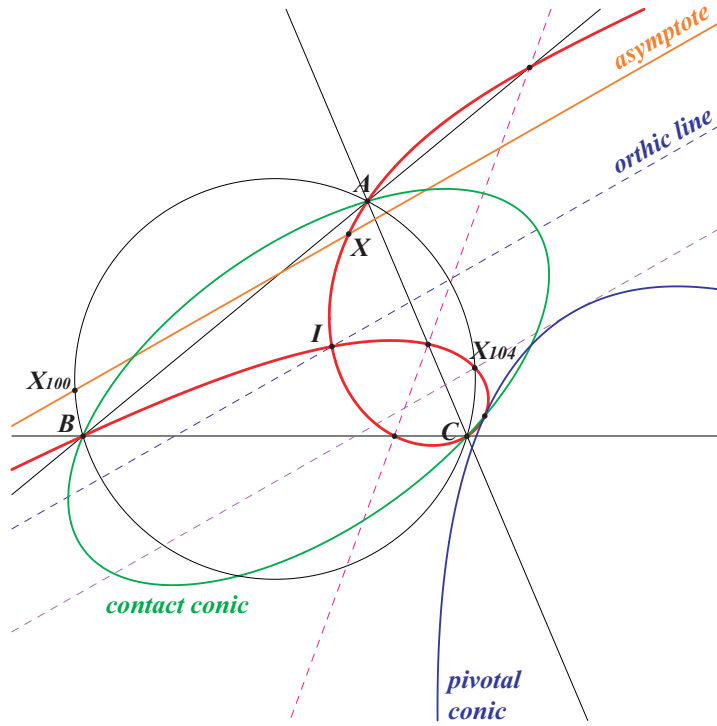


Figure 2: The strophoid $\mathcal{K}_1(X_{513}) = c\mathcal{K}(\#X_1, X_{905})$

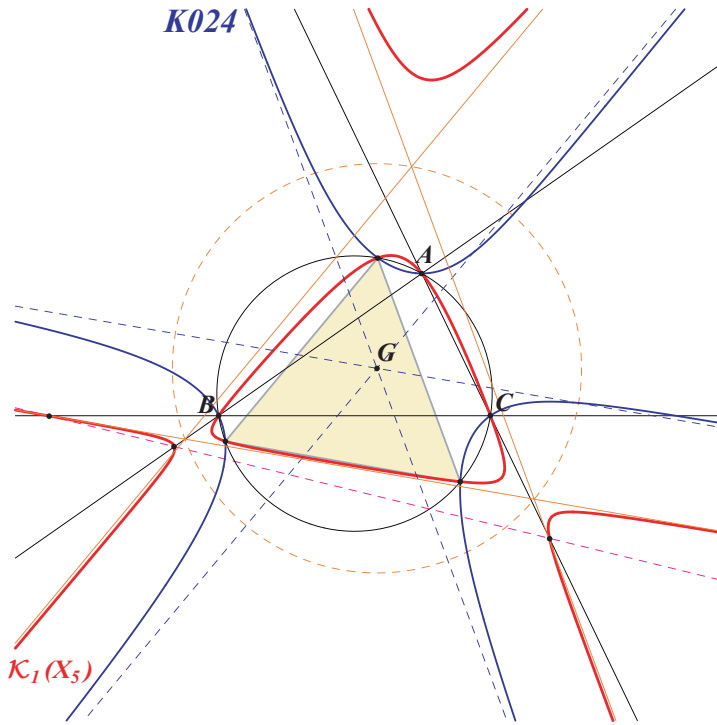


Figure 3: The equilateral cubic $\mathcal{K}_1(X_5)$

1.6 Central cubics

Following [3], §3.4.2, $\mathcal{K}_1(P)$ is a central cubic if and only if its center is its singular focus therefore if and only if F is one of the vertices of the circumnormal triangle, these points on the McCay cubic **K003**.

Thus, there are exactly three central $\mathcal{K}_1(P)$ when P is one of the infinite points of the Kjp cubic **K024** i.e. one of the infinite points of the sidelines of the Morley triangle. These three cubics are the only $\mathcal{K}^{++}$ of the net i.e. cubics with three asymptotes concurring on the cubic.

See figure 4.

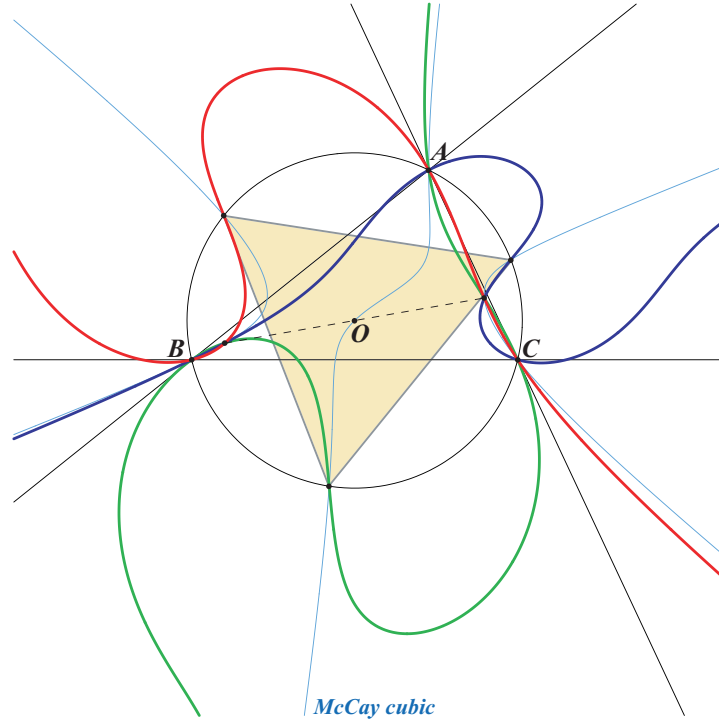


Figure 4: The three central $\mathcal{K}_1(P)$

2 The net $\mathcal{K}_2(P)$

2.1 Generalities

With the same notations as those in the previous section, another computation gives

Theorem 2. *The radical axis of the circumcircle ($\mathcal{O}$) of ABC and the pedal circle of a point M is perpendicular to the line passing through M and a fixed point P if and only if M lies on a circum-cubic $\mathcal{K}_2(P)$.*

This cubic $\mathcal{K}_2(P)$ has barycentric equation :

$$4\Delta^2 \sum p x (c^2 y^2 - b^2 z^2) + (a^2 yz + b^2 zx + c^2 xy) \sum (c^2 S_C q - b^2 S_B r) x = 0 \quad (2)$$

where Δ is the area of ABC , or equivalently

$$\sum [-S_B S_C p + a^2 S_A (q + r)] x (c^2 y^2 - b^2 z^2) + (x + y + z) \sum b^2 c^2 p x (S_B y - S_C z) = 0 \quad (3)$$

Properties of $\mathcal{K}_2(P)$

$\mathcal{K}_2(P)$ passes through P , the four foci of the inscribed conic with center O (these foci lie on the McCay cubic). It follows that these cubics form another net of cubics.

$\mathcal{K}_2(P)$ meets the sidelines of ABC at three points U, V, W such that U is the intersection of BC with the line through P and the antipode A' of A on $(\mathcal{O})$. V and W are defined in the same way.

Equation (2) shows that $\mathcal{K}_2(P)$ meets $(\mathcal{O})$ at the same points as $p\mathcal{K}(X_6, P)$, the isogonal pivotal cubic with pivot P . The three remaining intersections of the two cubics are P and two other isogonal conjugate points Q, Q^* on the line OP . These two points are the last common points of $\mathcal{K}_2(P)$ and the McCay cubic.

Equation (3) shows that $\mathcal{K}_2(P)$ meets $\mathcal{L}^\infty$ at the same points as $p\mathcal{K}(X_6, P')$, the isogonal pivotal cubic with pivot the reflection P' of P in O . The six remaining intersections of the two

cubics lie on the rectangular circum-hyperbola $\mathcal{H}(P)$ passing through P^* i.e. the conic that is the isogonal transform of the line OP . These points are A, B, C, Q, Q^* and P'^* , the isogonal conjugate of the reflection of P in O and the tangential of P' in $p\mathcal{K}(X_6, P')$. See figure 5.

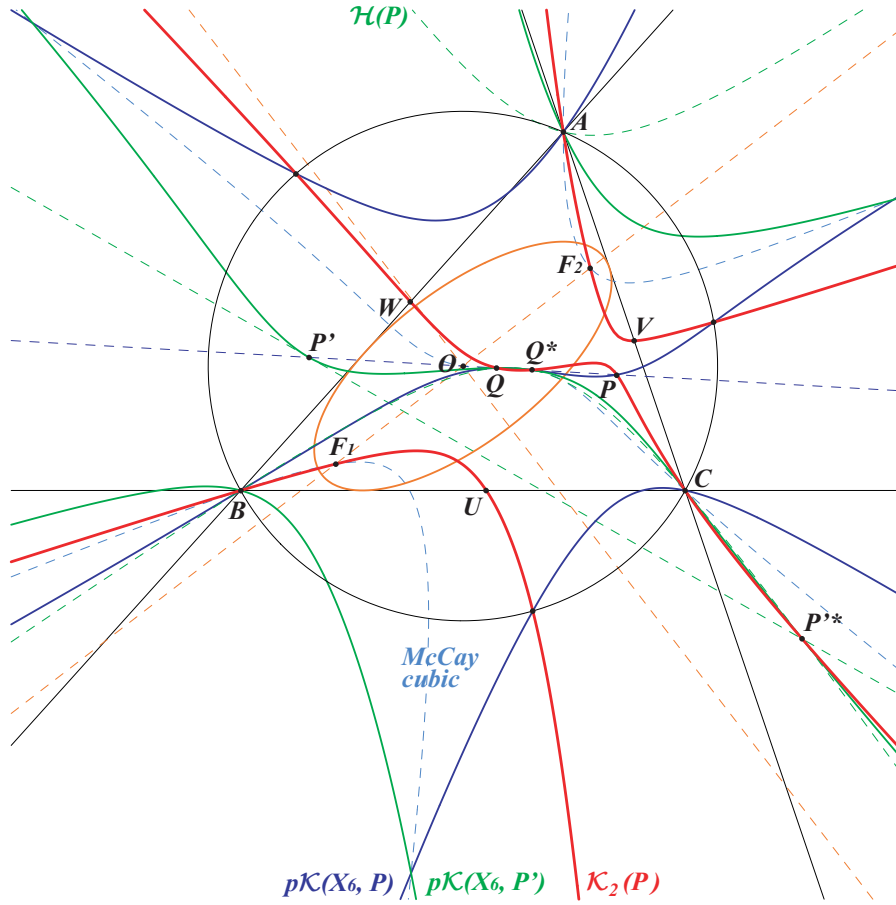


Figure 5: $\mathcal{K}_2(P)$ with $p\mathcal{K}(X_6, P)$ and $p\mathcal{K}(X_6, P')$

Construction of $K_2(P)$

Let l be a variable line passing through P and l' its reflection in O . The circum-conic isogonal transform of l' meets l at two points M, N on $K_2(P)$.

2.2 Special $K_2(P)$ of the net

1. U, V, W are collinear if and only if either :
 - P lies on $\mathcal{L}^\infty$ in which case $K_2(P)$ is an isogonal focal non-pivotal isocubic with root R on the trilinear polar of X_{69} , the isotomic conjugate of H . It is not surprising to see that, in this case, $K_2(P) = K_1(Q)$ where Q is the infinite point of the direction orthogonal to that defined by P .
 - P lies on $(\mathcal{O})$ in which case $K_2(P)$ is also a non-pivotal isocubic whose pole Ω is the barycentric product of P and the infinite point of the Simson line $\mathcal{S}(P)$ of P and whose root R is the barycentric quotient of P and H . R lies on the circum-conic with perspector O and the line PR contains X_{110} . The trilinear polar UVW of R contains O . Ω lies on the cubic **K162**, the isogonal transform of the Simson cubic **K010**.

This cubic contains P on $(\mathcal{O})$ and the infinite point of $\mathcal{S}(P)$. The two other points E_1, E_2 on $(\mathcal{O})$ are its intersections with the parallel at O to $\mathcal{S}(P)$. The two other points D_1, D_2 on $\mathcal{S}(P)$ are its intersections with the axes of the inscribed conic with center O . Recall that $\mathcal{K}_2(P)$ contains the four foci of this conic. See figure 6.

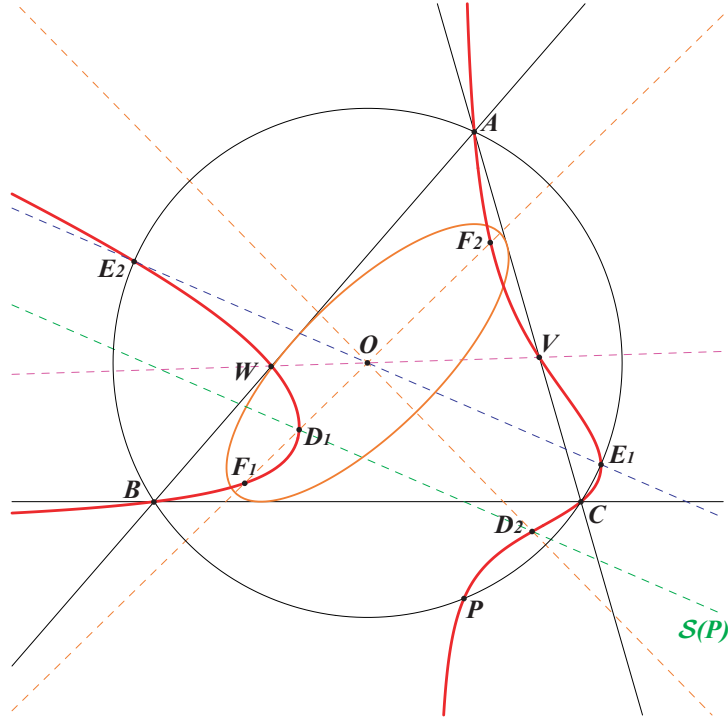


Figure 6: $\mathcal{K}_2(P)$ with P on $(\mathcal{O})$

2. The triangles UVW and ABC are in perspective if and only if P lies on the Darboux cubic **K004** and the perspector is a point on the Lucas cubic **K007**. In this case, the tangents at A, B, C to $\mathcal{K}_2(P)$ concur at a point X on the cubic $p\mathcal{K}(X_{2207}, X_4)$ we will call the Clawson cubic **K445**. See figure 7.

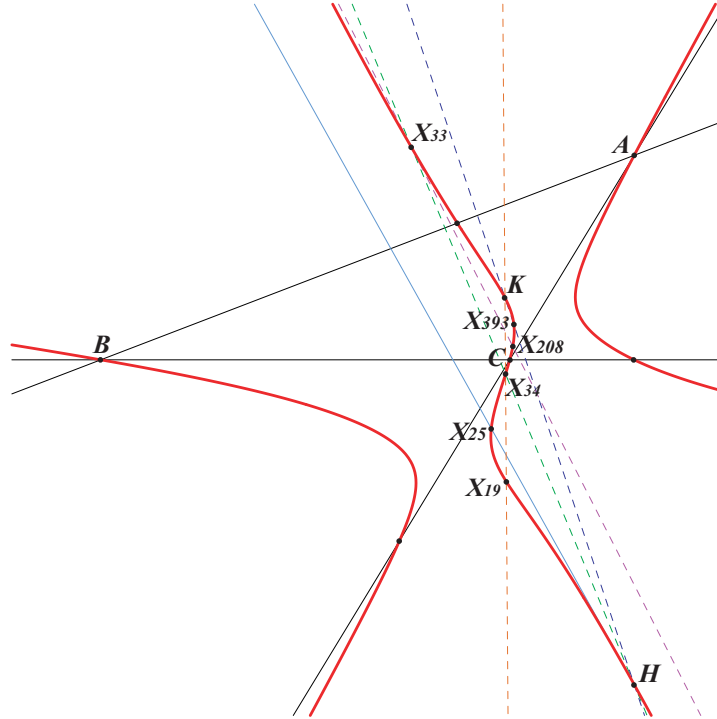


Figure 7: The Clawson cubic **K445**

K445 passes through $X_4, X_6, X_{19}, X_{25}, X_{33}, X_{34}, X_{64}, X_{208}, X_{393}$. The pole X_{2207} is the barycentric square of the Clawson point X_{19} hence the cubic also contains the extraversions of X_{19} . More generally, since this cubic is a strong cubic, it contains the extraversions of

each of its weak points, in particular X_{33} , X_{34} , X_{208} .

The isogonal transform of **K445** is **K099**, the Darboux perspector cubic.

The following table gives a selection of P on **K004** and the corresponding point X on **K445**.

P	X_1	X_3	X_4	X_{20}	X_{40}	X_{64}	X_{84}	X_{1490}	X_{1498}^*
X	X_{19}	X_4	X_{25}	X_6	X_{33}	X_{393}	X_{34}	X_{208}	X_{64}

Apart the McCay cubic **K003**, one of the most remarkable such cubics is $\mathcal{K}_2(X_{20})$, a cubic already met in [5] under the notation $\mathcal{K}(X_3)$ and catalogued **K443** in [4]. Recall that it is a central cubic passing the center of the inconic with center O , its foci, its contacts U , V , W with the sidelines of ABC and obviously their reflections in O . The tangents at A , B , C concur at the Lemoine point K . The asymptotes of $\mathcal{K}_2(X_{20})$ are parallel to those of the Orthocubic **K006**. See figure 8.

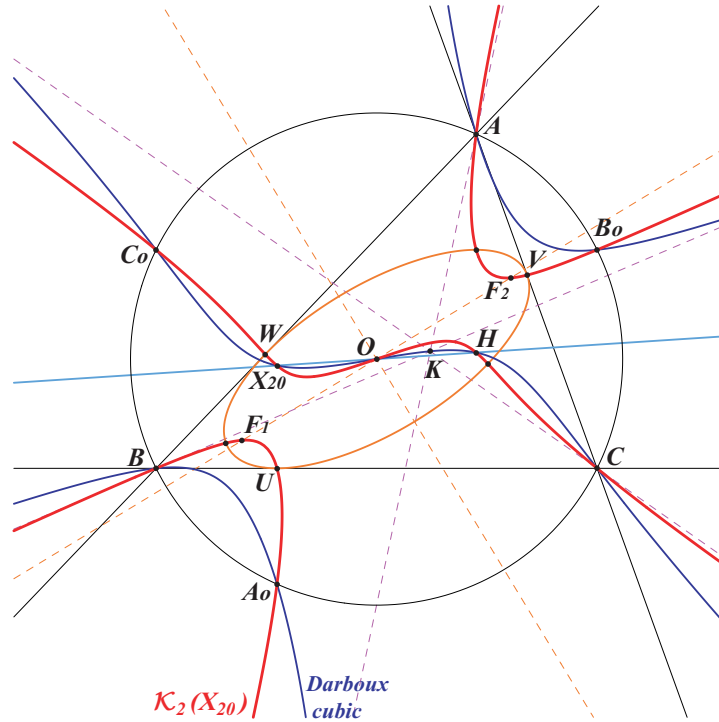


Figure 8: $\mathcal{K}_2(X_{20})$ or **K443**

$\mathcal{K}_2(X_{20})$ belongs to a pencil of central cubics generated by **K004** and the union of $(\mathcal{O})$ and the Euler line. This pencil also contains **K047**, **K080**.

3. $\mathcal{K}_2(P)$ is a $\mathcal{K}_0$ if and only if P lies on the Brocard axis OK .

These cubics form a pencil which contains :

- only one pivotal cubic $p\mathcal{K}$, the McCay cubic **K003** obtained when $P = O$, which is also the only equilateral cubic of the pencil,
- only one isogonal focal non-pivotal isocubic $n\mathcal{K}_0$, the third Brocard cubic **K019** obtained when $P = X_{511}$.

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