On the Thomson Triangle

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Abstract

The Thomson cubic meets the circumcircle at A, B, C and three other points Q_1, Q_2, Q_3 which are the vertices of the Thomson triangle $\mathcal{T}$. We investigate some properties of this triangle and, in particular, its connexion with the equiareality center X_{5373} related with the Steinhaus' problem.

1 Definition and properties of the Thomson triangle

1.1 Definition of $\mathcal{T}$

Let K002 be the Thomson cubic i.e. the isogonal pivotal cubic with pivot the centroid G of the reference triangle ABC. It is the locus of point M such that G, M and its isogonal conjugate M^* are collinear. This cubic has numerous properties which we will not consider here. See [2] for further details.

K002 meets the circumcircle (O) of ABC again at three (always real) points Q_1, Q_2, Q_3 which are the vertices of a triangle $\mathcal{T}$ we shall call the Thomson triangle.

Most of the results in this paper were obtained through manipulations of symmetric functions of the roots of third degree polynomials. In particular, the (barycentric) equation of the cubic that is the union of the sidelines of $\mathcal{T}$ is

$$\sum_{\text{cyclic}} \left[a^4yz(-x+2y+2z) - 2b^2c^2x(x-y)(x-z)\right] = 2\left(\sum_{\text{cyclic}} b^2c^2x\right)\left(\sum_{\text{cyclic}}(x^2-2yz)\right)$$

in which we recognize the equations of the Lemoine axis and the Steiner inellipse (S) namely :

$$\sum_{\text{cyclic}} b^2c^2x = 0 \quad \text{and} \quad \sum_{\text{cyclic}}(x^2-2yz) = 0.$$

The left-hand member of the equation of $\mathcal{T}$ represents a cubic curve which must be tritangent to (S).

This will give a good number of elements of $\mathcal{T}$ with an extensive use of the projective properties of a general cubic curve.

1.2 Some usual centers in $\mathcal{T}$

The following Table 1 gives a selection of several centers in ABC and their counterpart in the Thomson triangle $\mathcal{T}$. See the bottom of the page K002 in [2] for more.

Table 1: A selection of usual centers in $\mathcal{T}$

a center in ABC	X_1	X_2	X_3	X_4	X_5	X_6	X_{74}	X_{3146}
its counterpart in $\mathcal{T}$	X_{5373}	X_{3524}	X_3	X_2	X_{549}	X_{5646}	X_{110}	X_4

X_{5373} is the incenter of $\mathcal{T}$, see §4 below for further properties.

$K_{\mathcal{T}} = X_{5646}$ is the Lemoine point of $\mathcal{T}$, the intersection of the tangents at Q_1, Q_2, Q_3 to the Thomson cubic. This point lies on the lines X_2, X_{1350} and X_{64}, X_{631}. Note that the polar conic of $K_{\mathcal{T}}$ with respect to $\mathcal{T}$ is the circumcircle which turns out to be the circum-conic with perpsector $K_{\mathcal{T}}$ with respect to $\mathcal{T}$.

1.3 Miscellaneous properties

Here are some other properties of $\mathcal{T}$.

Property 1 : The Simson line of Q_i with respect to ABC is the reflection of Q_jQ_k about G and, similarly, the Simson line of A with respect to $\mathcal{T}$ is the reflection of BC about G.

Property 2 : Let M be a point on the circumcircle and f the fonction defined by $f(M) = MA.MB.MC$. f is obviously minimum when M is at A, B or C. It is (locally) maximum when M is one of the points Q_i.[1]

The product of these maxima is :

$$a^2b^2c^2\left(\frac{2R}{3}\right)^3, \quad R \text{ being the circumradius of } ABC.$$

It is known (see [7], p.15) that $MA.MB.MC = 4R^2\delta$ where δ is the distance from M to its Simson line. It follows that these points Q_i are those for which δ is (locally) maximum.

The product of these maxima is therefore :

$$\frac{a^2b^2c^2}{(6R)^3} = \frac{2\Delta^2}{27R}, \quad \Delta \text{ being the area of } ABC.$$

Property 3 : Jean-Pierre Ehrmann has shown that each altitude of $\mathcal{T}$ (i.e. each line X_2Q_i) is the common axis of the circumscribed and inscribed parabolas which have a maximal parameter. This is analogous to Property 2.

Figure 1 shows the three inscribed parabolas with foci the points Q_1, Q_2, Q_3 and vertices S_1, S_2, S_3.

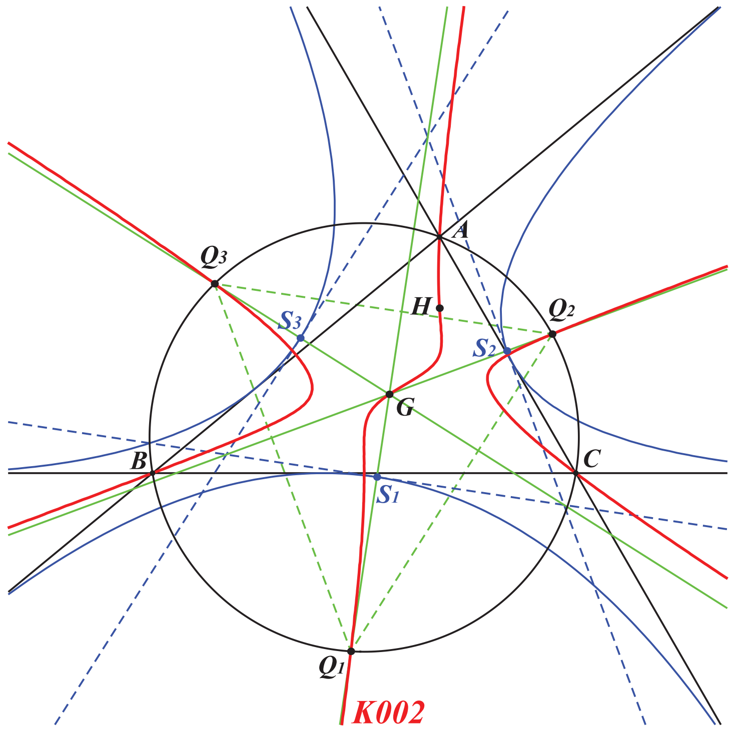

Figure 1: Three inscribed parabolas with maximal parameter

[1]This is discussed in the thread : Triangle des maxima, Les-Mathématiques.net, in French

Property 4 : Other properties.

1. $\overrightarrow{GQ_1} + \overrightarrow{GQ_2} + \overrightarrow{GQ_3} = 2\,\overrightarrow{GO}$.
2. $GQ_1.GQ_2.GQ_3 = \dfrac{1}{9} abc \cot\omega$, where ω is the Brocard angle.
3. $GQ_1^2 + GQ_2^2 + GQ_3^2 = 3R^2 + GO^2$ which can easily be generalized for any point M in the plane as follows :

   $MQ_1^2 + MQ_2^2 + MQ_3^2 = 3R^2 + MO^2 + 2\,\overrightarrow{GM}\cdot\overrightarrow{OM}$.
4. $Q_1Q_2^2 + Q_2Q_3^2 + Q_3Q_1^2 = 9R^2 - GO^2$.

1.4 Circumconics of the Thomson triangle

$\mathcal{T}$ is not constructible with ruler and compass but its vertices lie on several (easy to construct) conics (apart the circumcircle) and in particular on several rectangular hyperbolas forming a pencil and all containing G since it is the orthocenter of $\mathcal{T}$.

Note that the two triangles ABC and $\mathcal{T}$ share the same Euler line and obviously the same circumcenter O. The usual triangle centers on the Euler line in $\mathcal{T}$ are the images of those of ABC under the homothety with center O, ratio $1/3$. For example, the centroid of $\mathcal{T}$ is X_{3524}.

Naturally, the centers of these hyperbolas lie on the nine point circle of $\mathcal{T}$ which is the circle with center X_{549} (midpoint of GO) and radius $R/2$. This circle contains X_{2482}.

The equation of the rectangular hyperbola that contains the point $u : v : w$ is :

$$\sum_{\text{cyclic}} a^2[vw(-x+2y+2z)(y-z) + v(u-2v)z(x+y-2z) - w(u-2w)y(x-2y+z)] = 0.$$

Figure 2 shows the Jerabek hyperbola $J_{\mathcal{T}}$ of $\mathcal{T}$ (which is the rectangular hyperbola that contains the point O) with equation

$$\sum_{\text{cyclic}} b^2c^2(b^2-c^2)\,x\,(-2x+y+z) = 0.$$

Table 2 gives a selection of these rectangular hyperbolas.

Table 2: Rectangular hyperbolas passing through the vertices of $\mathcal{T}$

centers on the hyperbola apart G	remarks
X_3, X_6, X_{110}, X_{154}, X_{354}, X_{392}, X_{1201}, X_{2574}, X_{2575}, X_{3167}, see remark 1	$J_{\mathcal{T}}$
X_{511}, X_{512}, X_{574}, X_{805}, X_{3231}	
X_{55}, X_{513}, X_{517}, X_{672}, X_{901}, X_{1149}	
X_1, X_9, X_{100}, X_{165}, X_{3158}	
X_{30}, X_{230}, X_{476}, X_{523}	
X_{99}, X_{376}, X_{551}, X_{3413}, X_{3414}, see remark 2	center X_{2482}

Remark 1 : $J_{\mathcal{T}}$ is a member of the pencil of conics generated by the Jerabek and Stammler hyperbolas. Its center is X_{5642}.

$J_{\mathcal{T}}$ also contains X_{5544}, X_{5638}, X_{5639}, X_{5643}, X_{5644}, X_{5645}, X_{5646}, X_{5648}, X_{5652}, X_{5653}, X_{5654}, X_{5655}, X_{5656}. These points were added to [6] in June 2014.

Remark 2 : This rectangular hyperbola is the reflection of the Kiepert hyperbola about G.

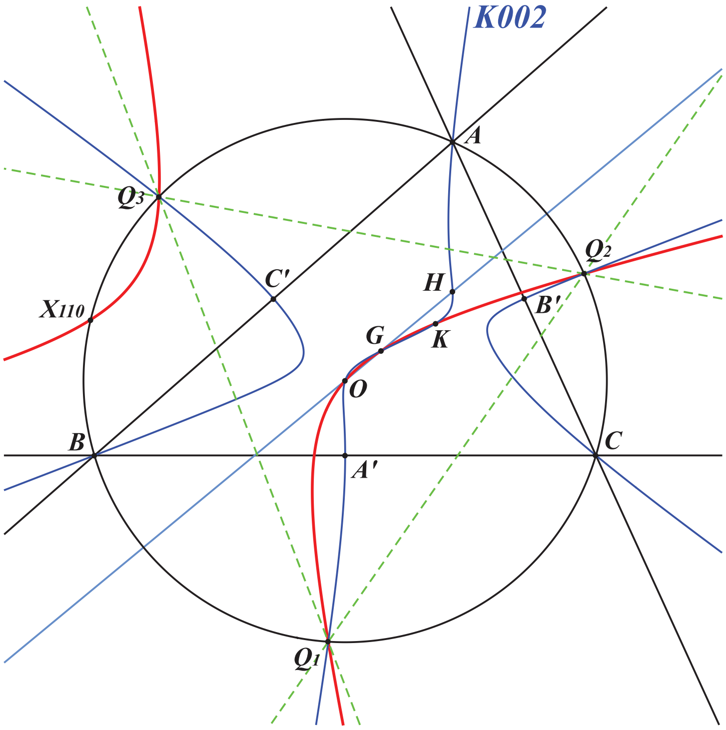

Figure 2: Jerabek hyperbola of $\mathcal{T}$

Remark 3 : $J_{\mathcal{T}}$ is connected with the cubics of the Euler pencil (see Table 27) in the following manner. Let P be the point on the Euler line such that $\overrightarrow{OP} = t\,\overrightarrow{OH}$ and let $\mathcal{K} = p\mathcal{K}(X_6, P)$ be the isogonal pivotal cubic with pivot P. There is one and only one point Q whose polar conic in $\mathcal{K}$ is a (possibly degenerate) circle and this point lies on $J_{\mathcal{T}}$.

The first coordinate of Q is :

$$a^2[3\,b^2c^2 + (a^2b^2 + a^2c^2 - b^4 - c^4)\,t - (a^2 - b^2)(a^2 - c^2)\,t^2].$$

The following table 3 gives Q for the most remarkable cubics of Euler pencil.

Table 3: $J_{\mathcal{T}}$ and the cubics of the Euler pencil

t	∞	1/3	0	-1	1/2	1	-1/3
P	X_{30}	X_2	X_3	X_{20}	X_5	X_4	X_{376}
Q	X_{110}	X_{5544}	X_2	X_3	X_{5643}	X_6	X_{5646}
$\mathcal{K}$	K001	K002	K003	K004	K005	K006	K243

Remark 4 : In a similar way, $J_{\mathcal{T}}$ is connected with the cubics $n\mathcal{K}_0(X_6, R)$ where the root R is a point on the line $GK = X_2X_6$. Indeed, for any such point R, there is also one and only one point Q whose polar conic in $n\mathcal{K}_0(X_6, R)$ is a (possibly degenerate) circle and this point lies on $J_{\mathcal{T}}$.

When $R = X_2$ and $R = X_6$, we obtain the cubics K082 and K024 with corresponding points $Q = X_6$ and $Q = X_2$ respectively but, in both cases, the polar conic splits into the line at infinity and another line. Recall that these two cubics are $\mathcal{K}^+$ i.e. have concurring asymptotes. There is actually a third $\mathcal{K}^+$ in the pencil but its root is complicated and unlisted in ETC.

When $R = X_{230}$, the cubic is circular, namely K189, and then $Q = X_{110}$ is its singular focus. Another example is obtained with $R = X_{385}$ and the cubic K017. The corresponding point Q is the second intersection (apart G) of the line X_2X_{694} with $J_{\mathcal{T}}$, SEARCH = 7.71122335962109.

Remark 5 : Conversely, if Q is taken on $J_{\mathcal{T}}$, one can find P on the Euler line and R on the line GK such that the polar conics of Q in $p\mathcal{K}(X_6, P)$ and $n\mathcal{K}_0(X_6, R)$ are both circles.

If P' (on the Euler line) is the isogonal conjugate of Q with respect to the Thomson triangle (see §3 below), then P is its homothetic under $h(O, -1/3)$. Let $\mathcal{E}$ be the ellipse passing through X_4, X_{32}, X_{7736} and tangent at these points to the Euler line, the Brocard axis, the line GK respectively. $\mathcal{E}$ also passes through X_{1506}, X_{6781}. The tangent at P to $\mathcal{E}$ which is not the Euler line meets the line GK at the requested point R. See figure 3.

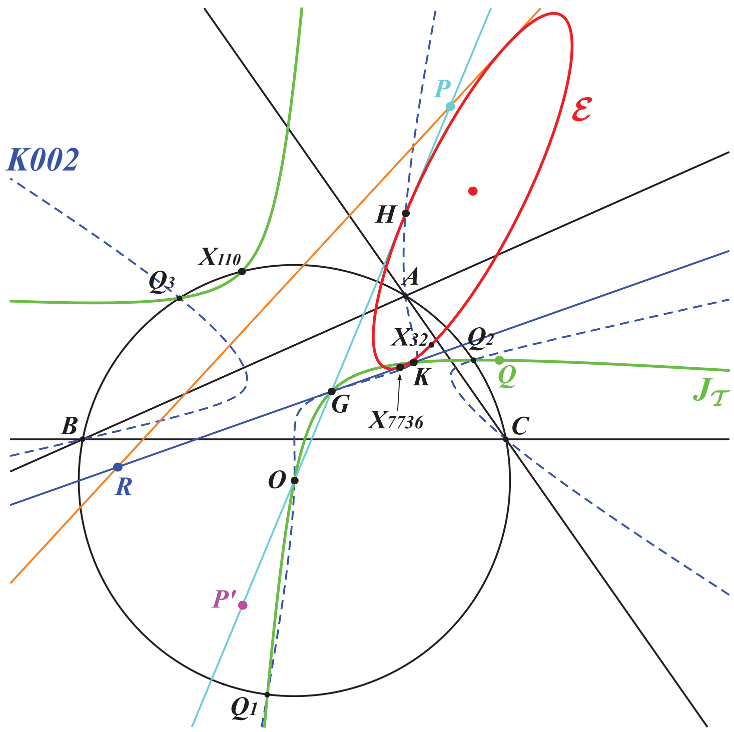

Figure 3: $J_{\mathcal{T}}$ and circular polar conics in $p\mathcal{K}(X_6, P)$ and $n\mathcal{K}_0(X_6, R)$

For a given point Q on $J_{\mathcal{T}}$, $p\mathcal{K}(X_6, P)$ and $n\mathcal{K}_0(X_6, R)$ generate a pencil of cubics and the polar conic of Q in any cubic is a (possibly degenerated) circle.

Examples :

– with $Q = X_{110}$, the cubics are the Neuberg cubic K001 and X189, both circular with focus X_{110}.

– with $Q = X_2$, the cubics are the McCay cubic K003 and the Kjp cubic X024, both stelloids with radial center X_2.

– with $Q = X_6$, the cubics are K006 and K082.

1.5 Inconics of the Thomson triangle

Any conic inscribed in $\mathcal{T}$ is the poloconic of a line with respect to $\mathcal{T}$. On the other hand, since the triangles ABC and $\mathcal{T}$ are both inscribed in (O), they must circumscribe a same conic which is the Steiner inellipse (S) of ABC. This is a classical property of poristic triangles.

The contacts of (S) with the sidelines of $\mathcal{T}$ are the feet R_1, R_2, R_3 of the normals drawn from the Lemoine point K to (S), the fourth foot being X_{115}, the center of the Kiepert hyperbola. These points R_1, R_2, R_3 also lie on the Apollonius rectangular hyperbola of K with respect to (S). This latter hyperbola is homothetic to the Kiepert hyperbola and passes through X_2, X_4, X_6, X_{39}, X_{115}, X_{1640}.

Note that R_i is the barycentric square of the isogonal conjugate Q_i^* of Q_i and also Q_jQ_k is the trilinear polar of Q_i^*.

These three lines Q_iR_i concur at the intersection X_{QR} of the lines X_6, X_{376} and X_{39}, X_{631}, a point with first barycentric coordinate :

$$3a^2(a^2 + 4b^2 + 4c^2) + (b^2 - c^2)^2$$

and SEARCH = 1.3852076515.

This point X_{QR} does not lie on the Thomson cubic K002 hence K002 is not a pivotal cubic with respect to $\mathcal{T}$ but "only" a $ps\mathcal{K}$ (see [3]) with pseudo-pivot X_{QR} and pseudo-isopivot $K_{\mathcal{T}}$, the Lemoine point of $\mathcal{T}$.

Figure 4 shows the Steiner inellipse inscribed in both triangles ABC and $\mathcal{T}$.

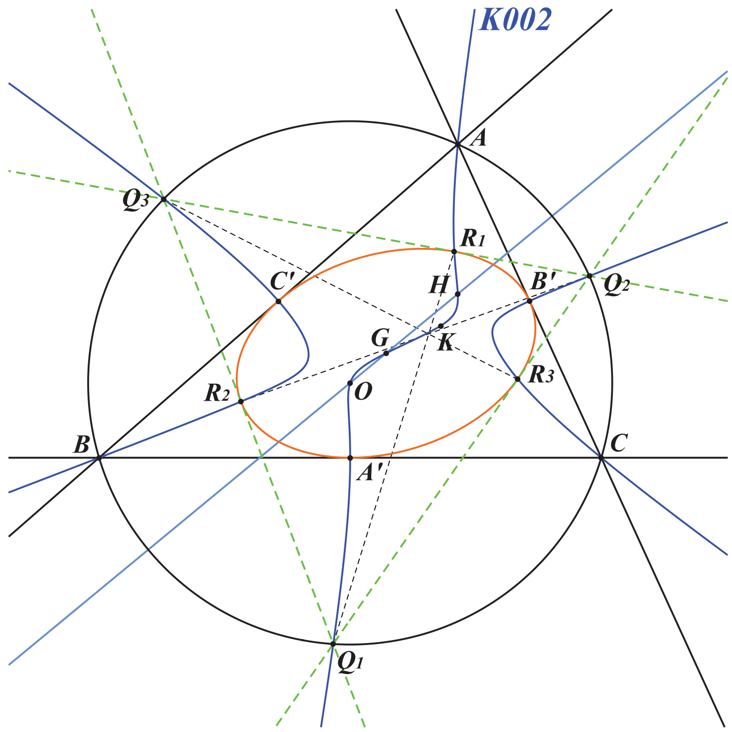

Figure 4: The Steiner inellipse is inscribed in $\mathcal{T}$

Furthermore, the parabola with focus X_{74} and directrix the Euler line is the Kiepert parabola of $\mathcal{T}$. It is the reflection about O of the Kiepert parabola (of ABC).

1.6 Diagonal conics of the Thomson triangle

A conic is said to be diagonal with respect to a certain triangle when the triangle is self-polar in the conic i.e. the polar line of one vertex of the triangle is the opposite sideline.

A computation gives the equation of the polar circle of $\mathcal{T}$ which turns out to be the circle with center G (the orthocenter of $\mathcal{T}$) and radius the square root of $-(a^2+b^2+c^2)/18$. This shows that this circle is always imaginary hence that $\mathcal{T}$ is always an acute angled triangle.

The same technique gives the diagonal rectangular hyperbolas which must contain the in/excenters of $\mathcal{T}$. These form a pencil and their centers lie on the circumcircle.

The equation of that passing through a given point is rather tedious. We shall only give three examples which are more than enough to construct these in/excenters. See table 4.

Table 4: Diagonal rectangular hyperbolas with respect to $\mathcal{T}$

centers on the hyperbola	center of the hyperbola
X_3, X_{40}, X_{64}, X_{1350}, X_{2574}, X_{2575}	X_{74}, see below
X_2, X_{3413}, X_{3414}	X_{98}
X_{30}, X_{523}, X_{549}	X_{477}

Figure 5 shows the two diagonal rectangular hyperbolas passing through O (plain red curve) and G (dashed red curve). Their asymptotes are parallel to those of the Jerabek and Kiepert hyperbolas respectively. Note that the former hyperbola is the Stammler hyperbola of $\mathcal{T}$. It is the reflection of the Stammler hyperbola (of ABC) about O.

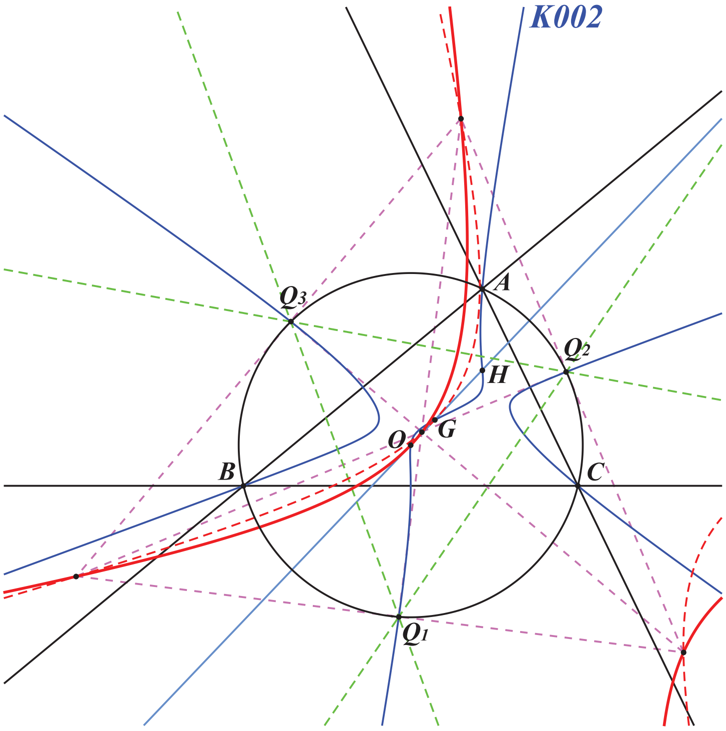

Figure 5: Two diagonal rectangular hyperbolas

In §4 we shall see the connexion of these hyperbolas with the equiareality center X_{5373}.

2 Cubics related with the Thomson triangle

In this section, we characterize the cubics – apart the Thomson cubic K002 – that pass through the vertices of $\mathcal{T}$.

2.1 Other pivotal cubics passing through the vertices of $\mathcal{T}$

We simply recall several results already mentioned (and generalized) in [4].

Proposition 1 *The pivotal cubic $p\mathcal{K}(\Omega, P)$ passes through the vertices of the Thomson triangle if and only if :*

- *its pole Ω lies on* K346 $= p\mathcal{K}(X_{1501}, X_6)$,
  passing through X_i for $i = 6, 25, 31, 32, 41, 184, 604, 2199, 3172, 7118$.
- *its pivot P lies on the Thomson cubic* K002,
- *its isopivot P^* lies on* K172 $= p\mathcal{K}(X_{32}, X_3)$,
  passing through X_i for $i = 3, 6, 25, 55, 56, 64, 154, 198, 1033, 1035, 1436, 7037$.

Note that each cubic contains X_6 and then also P/X_6 (which is a point on K172) and its Ω−isoconjugate $(P/X_6)^*$ (which is a point on the Darboux cubic K004).

When the pivot P is chosen on K002, the isopivot P^* and the pole Ω are the barycentric products of X_6 by aP (the anticomplement of P) and X_2/P (center of the circum-conic with perspector P) respectively.

Furthermore, if P and P' are two isogonal conjugate points on the Thomson cubic K002, then
– the two corresponding poles are two points of K346 collinear with X_{25},
– the two corresponding isopivots are two points of K172 collinear with X_6,
– the two corresponding points $(P/X_6)^*$ are two points of K004 collinear with X_{20}, hence isogonal conjugates in ABC.

Table 5 gives a selection of cubics according to their pivot P on the Thomson cubic K002.

Table 5: Pivotal cubics passing through the vertices of $\mathcal{T}$

P	Ω (X_i or SEARCH)	cubic or X_i on the cubic
X_1	X_{41}	K761
X_2	X_6	K002
X_3	X_{32}	K172
X_4	X_{3172}	X_4, X_6, X_{20}, X_{25}, X_{154}, X_{1249}
X_6	X_{184}	K167
X_9	X_{31}	K760
X_{57}	X_{2199}	X_6, X_{40}, X_{56}, X_{57}, X_{198}, X_{223}
X_{223}	X_{604}	X_6, X_{57}, X_{223}, X_{266}, X_{1035}, X_{1436}, X_{3345}
X_{282}	0.3666241407629	X_6, X_{282}, X_{1035}, X_{1436}, X_{1490}
X_{1073}	0.6990940852287	X_6, X_{64}, X_{1033}, X_{1073}, X_{1498}
X_{1249}	X_{25}	X_4, X_6, X_{64}, X_{1033}, X_{1249}, X_{3346}

2.2 Non-pivotal cubics passing through the vertices of $\mathcal{T}$

An easy computation shows that one can find a proper non-pivotal isocubic $n\mathcal{K}$ passing through the vertices of the Thomson triangle if and only if it is a $n\mathcal{K}_0$ i.e. a cubic without term in xyz. Indeed, the presence of a term in xyz yields to a cubic that must decompose into the circumcircle and a line which is its isoconjugate and therefore the trilinear polar of its root.

Let us then consider a non-pivotal isocubic $n\mathcal{K}_0(\Omega, P)$.

Proposition 2 *The cubic $n\mathcal{K}_0(\Omega, P)$ passes through the vertices of the Thomson triangle if and only if :*

- *its pole Ω lies on the Lemoine axis i.e. the trilinear polar of X_6,*
- *its root P lies on the line at infinity.*

The line Ω, P envelopes the parabola with focus X_{110}, directrix the perpendicular at X_{23} to the Euler line or, equivalently, the polar line of G in the circumcircle.

Note that each cubic contains X_6 again. Figure 6 below presents two of these cubics $n\mathcal{K}_0$ namely :

– K624 $= n\mathcal{K}_0^+(X_{512}, X_{30})$ which has three real asymptotes councurring at G. It contains X_6, X_{523}, X_{2574}, X_{2575}.

– K625 $= n\mathcal{K}_0(X_{351}, X_{542})$ passing through X_6, X_{187}, X_{511}, X_{523}, X_{690}.

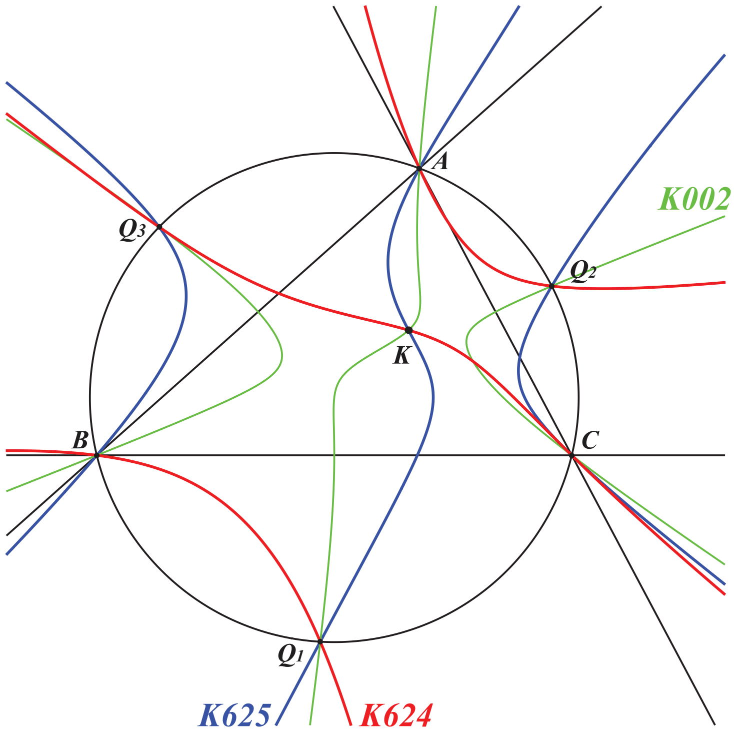

Figure 6: K624 and K625 in the Thomson triangle

2.3 Pseudo-pivotal cubics passing through the vertices of $\mathcal{T}$

Recall that a pseudo-pivotal cubic $ps\mathcal{K}$ is a circum-cubic such that equivalently :

- the other intersections with the sidelines of ABC form a triangle perspective (at P) with ABC,
- the tangents at A, B, C concur (at Q).

In this case, P, Q, $\Omega = P \times Q$ are called pseudo-pivot, pseudo-isopivot, pseudo-pole respectively. When P (and then Q) lies on the cubic, it becomes a pivotal cubic. See [3] for more informations.

An easy computation gives

Proposition 3 *A pseudo-pivotal cubic $ps\mathcal{K}(\Omega, P)$ passes through the vertices of the Thomson triangle if and only if :*

- *its pseudo-pole* Ω *is* $X_6 \times G/P$ *where* G/P *is the center of the circum-conic with perspector* P*,*

- *its pseudo-pivot* P *is* $G/(\Omega \times X_{76})$*.*

It follows that, for given Ω or P, there is one and only one such cubic.
For $P = u : v : w$, its equation is :

$$\sum_{\text{cyclic}} a^2yz[u(-u+v+w)(wy-vz)+vw(v-w)x] = 0.$$

Naturally, for any P on the Thomson cubic, this $ps\mathcal{K}$ becomes a $p\mathcal{K}$ and passes through K, P and P/K. In this case, its pole lies on K346. See §2.1.

Figure 7 shows $ps\mathcal{K}(X_{3167}, X_{69}, X_3)$ passing through X_3, X_{20}, X_{459}, X_{3167} and the vertices of the cevian triangle of X_{69}, its pseudo-pivot. The pseudo-isopivot is X_{3053}.

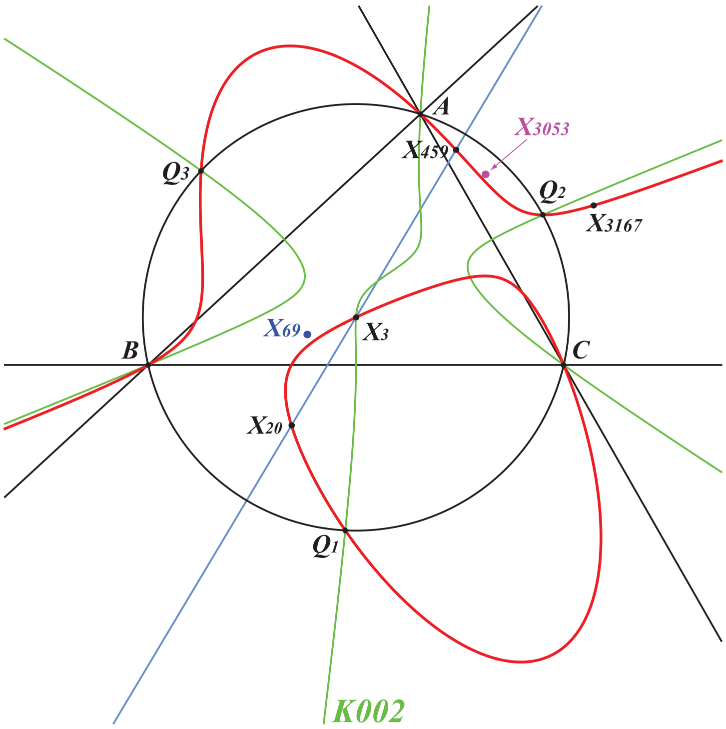

Figure 7: $ps\mathcal{K}(X_{3167}, X_{69}, X_3)$ in the Thomson triangle

2.4 Equilateral cubics passing through the vertices of $\mathcal{T}$

Recall that a cubic is said to be equilateral when it has three real asymptotes making 60° angles with one another. This occurs when the polar conics of the points at infinity are rectangular hyperbolas i.e. when the orthic line of the cubic (when it is defined i.e. when the asymptotes do not concur) is the line at infinity.

Proposition 4 *Any equilateral cubics passing through the vertices of* $\mathcal{T}$ *must contain the infinite points of the McCay cubic* K003.

This shows that all these equilateral cubics have nine common points (six on the circumcircle and three at infinity) and therefore belong to a same pencil obviously containing the decomposed cubic which is the union of the circumcircle and the line at infinity.

There is one and only one cubic with concurring asymptotes (at X_{5055}) in this pencil and this is K581 passing through X_2, X_3, X_4, X_{262}, the foci of the inscribed conic with center the midpoint X_{549} of GO. See figure 8.

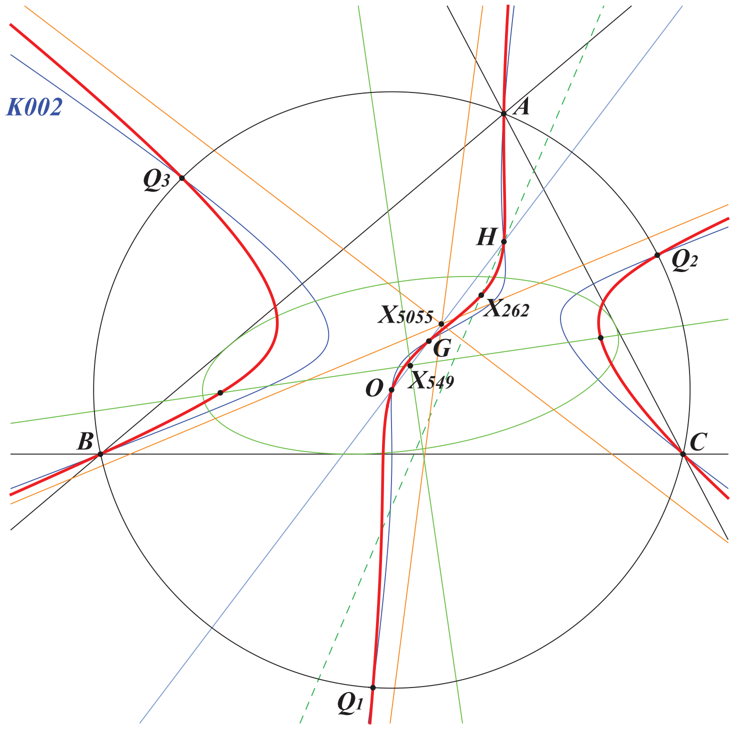

Figure 8: The equilateral cubic K581 in the Thomson triangle

2.5 Nodal cubics passing through the vertices of $\mathcal{T}$

Let P be a point not lying on a sideline of ABC nor on the circumcircle, this to avoid decomposed cubics.

After some computations, we have

Proposition 5 *There is a unique nodal cubic with node P passing through the vertices of the Thomson triangle. Moreover, there is always an isogonal pivotal cubic having the same points at infinity.*

For example, K280 and K297 are two nodal cubics with nodes G and K respectively. We observe that these two cubics are actually two $\mathcal{K}_0$ (without term in xyz) and that both contain the Lemoine point K. More generally, we have

Proposition 6 *The following assertions are equivalent :*

- *the nodal cubic with node P is a $\mathcal{K}_0$,*
- *the nodal cubic with node P contains K,*
- *P lies on the nodal quartic* Q090 which is the isogonal transform of the Stothers quintic Q012.

Q090 contains X_2, X_6, X_{15}, X_{16}, X_{55}, X_{385}, X_{672} and obviously the isogonal conjugates of all the points of Q012. Its equation is remarkably simple :

$$\sum_{\text{cyclic}} b^2c^2x^2[b^2(x-y)z - c^2(x-z)y] = 0.$$

The nodal tangents at P are perpendicular if and only if P lies on the circular quintic Q091 which is the locus of P such that P and its isogonal conjugates in both triangles ABC and $\mathcal{T}$ are collinear. See §3 below.

Q091 contains the vertices and the in/excenters of both triangles ABC and $\mathcal{T}$ (namely X_1, X_{5373} and their harmonic associates, see §4), O which is their common circumcenter, the infinite points of the McCay cubic K003, the four foci of the Steiner ellipse inscribed in both triangles.

The most remarkable corresponding cubic is probably K626, that obtained with $P = O$, which turns out to be the isogonal transform of K616. K626 passes through X_3, X_{25}, X_{1073}, X_{1384}, X_{1617}, X_{3167} and its nodal tangents are parallel to the asymptotes of the Jerabek hyperbola. See figure 9.

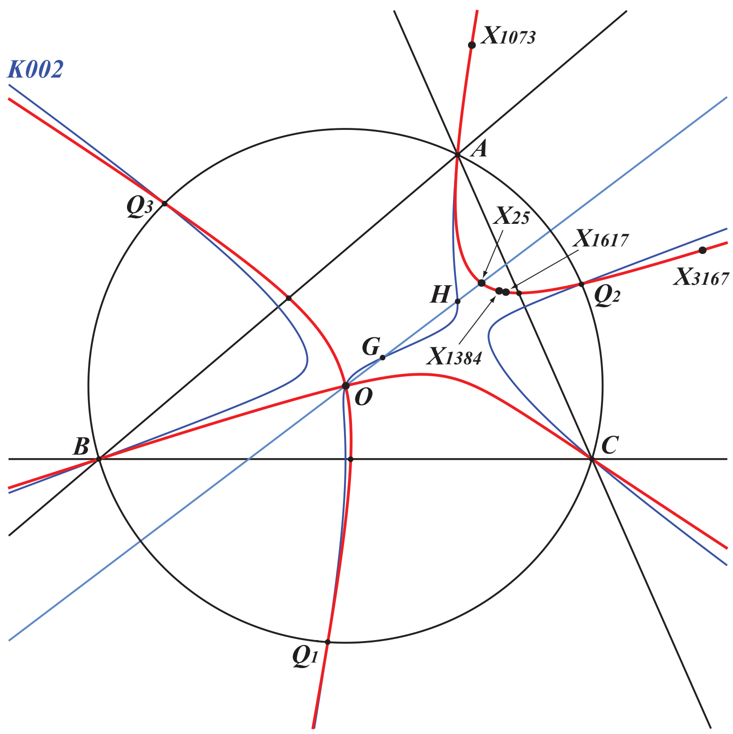

Figure 9: The nodal cubic K626 in the Thomson triangle

2.6 $sp\mathcal{K}$ cubics passing through the vertices of $\mathcal{T}$

This type of cubic is defined in CL055 of [2]. From the informations found there, we obtain

Proposition 7 *A cubic $sp\mathcal{K}(P,Q)$ passes through the vertices of the Thomson triangle if and only if Q is the midpoint of GP.*

This cubic, hereby denoted by $sp\mathcal{K}(P)$, is the locus of the common points of a variable line passing through G and the isogonal transform of its parallel at P.

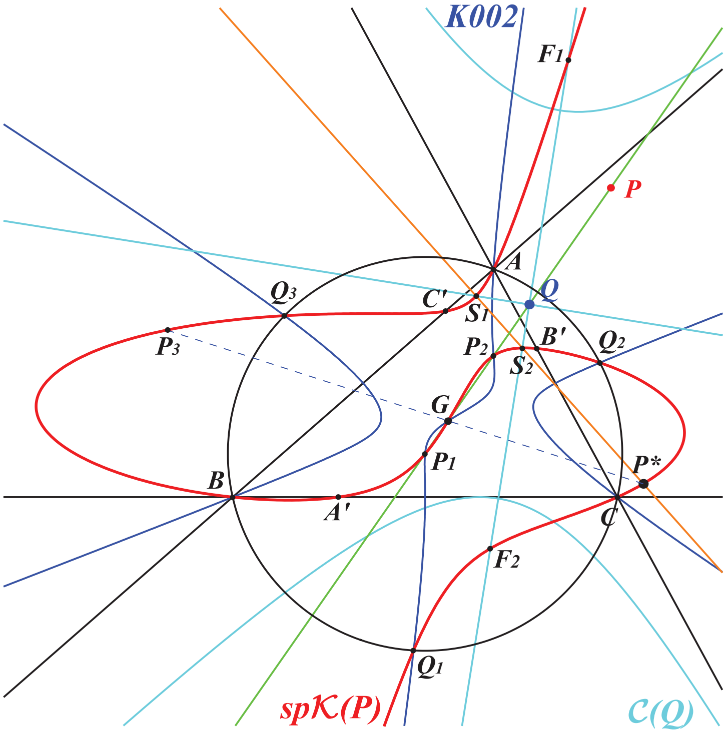

Figure 10: $sp\mathcal{K}$ cubics in the Thomson triangle

When P lies at infinity, $sp\mathcal{K}(P)$ splits into the circumcircle and the line GP^* where P^* is the isogonal conjugate of P. This is excluded in the sequel.

For any (finite) point P, the cubic $sp\mathcal{K}(P)$ contains (see Figure 10) :

1. A, B, C, G, Q_1, Q_2, Q_3 hence these cubics form a net.
2. P^*. The third point P_3 on the line GP^* also lies on the line passing through P and the isogonal conjugate of the infinite point of the line GP^*.
3. the infinite points of $p\mathcal{K}(X_6, P)$.
4. two (real or not) points P_1, P_2 on the line GP, on the circum-conic which is its isogonal transform hence on the Thomson cubic K002.
5. the foci of the inconic $\mathcal{C}(Q)$ with center Q.
6. the two other intersections S_1, S_2 with the axes of this inconic which are collinear with P^*.
7. A', B', C' on the sidelines of ABC and on the parallels at G to the lines AP, BP, CP respectively.

For some particular points P, we meet some special cubics again, those already mentioned in the previous paragraphs. For example, $sp\mathcal{K}(G)$ is K002, the only $p\mathcal{K}$ of the net.

Furthermore, every $sp\mathcal{K}(P)$ with P on
– K002 passes through P,
– the line GK is a $\mathcal{K}_0$,
– the Steiner ellipse (S) is a $n\mathcal{K}$.

It follows that there are two cubics $sp\mathcal{K}(P)$ which are $n\mathcal{K}_0$ obtained when P is one of the common points of line GK and (S), namely X_{6189} and X_{6190}, two antipodes on (S).

The corresponding cubics are $sp\mathcal{K}(X_{6189}) = n\mathcal{K}_0(X_{5639}, X_{3413})$ and $sp\mathcal{K}(X_{6190}) = n\mathcal{K}_0(X_{5638}, X_{3414})$.

Note that the trilinear polars $\mathcal{P}(X_{3413})$, $\mathcal{P}(X_{3414})$ of X_{3413}, X_{3414} – the infinite points of the Kiepert hyperbola – are parallel and meet the sidelines of $\mathcal{T}$ at three points lying on the cubics $n\mathcal{K}_0$. More precisely, $sp\mathcal{K}(X_{6189})$ meets the sidelines of ABC at U_1, V_1, W_1 lying on the trilinear polar of X_{3413} and the sidelines of $\mathcal{T}$ on the trilinear polar of X_{3414}. Hence, these two cubics are also $n\mathcal{K}s$ with respect to $\mathcal{T}$. See Figure 11.

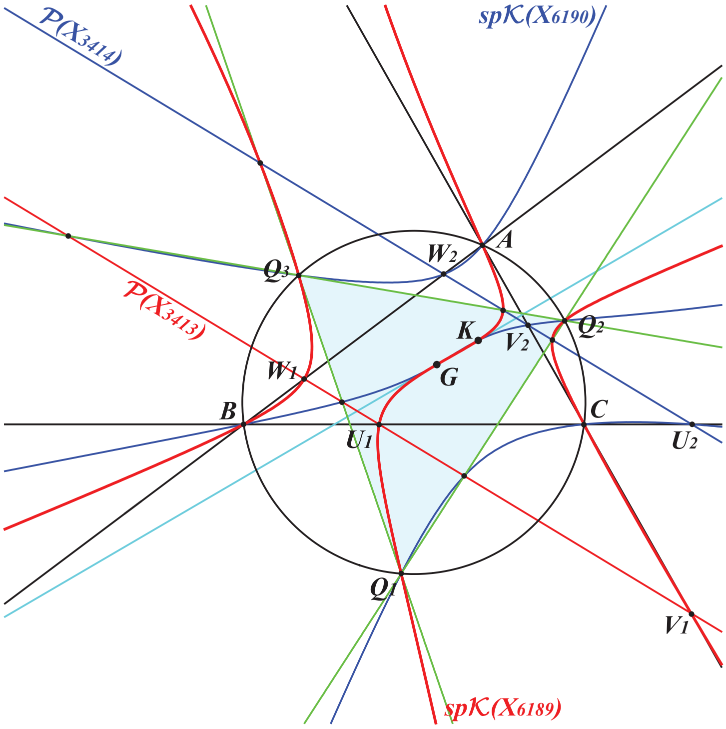

Figure 11: $sp\mathcal{K}(X_{6189})$ and $sp\mathcal{K}(X_{6190})$

3 Isogonal conjugation with respect to $\mathcal{T}$

Recall that the diagonal rectangular hyperbolas we met above form a pencil of conics passing through the in/excenters of $\mathcal{T}$ and obviously the diagonal triangle of these four points is $\mathcal{T}$. It follows that the intersection of the polar lines of a point M in any two of these hyperbolas concur at a same point that is the isogonal conjugate $M^*_{\mathcal{T}}$ of M with respect to $\mathcal{T}$.

With $M = x : y : z$, the first coordinate of $M^*_{\mathcal{T}}$ is :

$$a^2[2S_A(a^2yz + b^2zx + c^2xy) - b^2c^2\,x^2 + c^2S_C\,y^2 + b^2S_B\,z^2 - 4\Delta^2yz],$$

where Δ is the area of ABC.

Naturally, since ABC and $\mathcal{T}$ share the same circumcircle (O), the isogonal conjugates M^* and $M^*_{\mathcal{T}}$ of M lying on the line at infinity both lie on (O). Furthermore, these two points are antipodes on (O).

In particular, when M is the infinite point of the altitude AH of ABC, M^* is the antipode of A and $M^*_{\mathcal{T}}$ is A itself.

Randy Hutson observes that $M^*_{\mathcal{T}}$ is the centroid of the antipedal triangle of M^* (with respect to ABC).

3.1 Bi-isogonal conjugates

When we look for points M having the same isogonal conjugate in both triangles, we find that these point must lie on three focal cubics, each passing through one vertex of ABC (which is the singular focus) and the three vertices of $\mathcal{T}$. These cubics belong to a same pencil hence they must have six other common points which are the requested so-called bi-isogonal points.

These points are the two circular points at infinity and the four foci (two only are real) of the Steiner inellipse. This is explained by the fact that the two triangles circumscribe this latter ellipse.

3.2 Isogonal conjugates of some usual centers

The following table gives a selection of several centers in ABC and their isogonal conjugates with respect to $\mathcal{T}$.

Table 6: A table of usual isogonal conjugates

a center in ABC	its isogonal in $\mathcal{T}$
X_1	X_{165}
X_2	X_3
X_3	X_2
X_4	X_{154}
X_5	X_{6030}
X_6	X_{376}
X_{20}	X_{3167}
X_{30}	X_{110}
X_{376}	X_6

Remark 1 : all the points on the Euler line have their isogonal conjugates on the Jerabek hyperbola of $\mathcal{T}$. See below.

Remark 2 : the isogonal conjugate of X_5 in $\mathcal{T}$ is the point with abscissa 5/3 in X_6, X_{1176} and with SEARCH = 85.5364033750526. This point is now X_{6030} in ETC.

3.3 Isogonal conjugates of some usual lines and related conics

Since the circumcenter of $\mathcal{T}$ is O, it is clear that the isogonal conjugate (with respect to $\mathcal{T}$) of any line L_O passing through O is a rectangular circum-hyperbola in $\mathcal{T}$ that must contain G, the orthocenter of $\mathcal{T}$. In particular, the isogonal conjugate of the Euler line is the Jerabek hyperbola of $\mathcal{T}$ as already said.

The two rectangular hyperbolas obtained from L_O by isogonal conjugation in triangles ABC and $\mathcal{T}$ have parallel asymptotes. They must meet at two other finite points collinear with K that lie on the Thomson cubic. It follows that these points are $G-$Ceva conjugate points.

Figure 12 shows the line L_O passing through X_{399} on the Stammler hyperbola and the two related rectangular hyperbolas.

The tangent at O to the Stammler hyperbola is the Euler line in which case the two hyperbolas are the Jerabek hyperbolas of ABC and $\mathcal{T}$ passing through O and K.

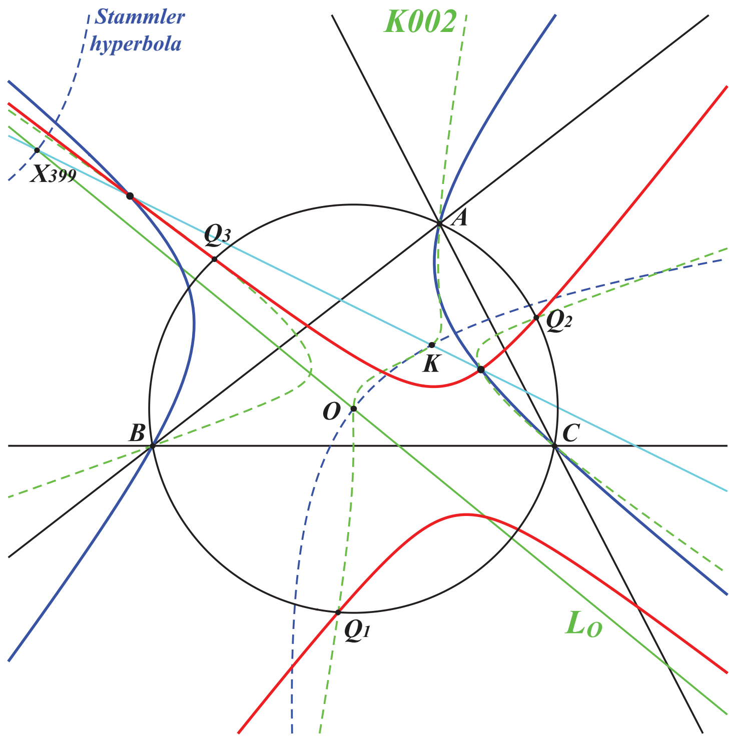

Figure 12: Intersection of two isogonal transforms of a line through O

Naturally, when L_O passes through one focus of the Steiner inellipse then the two hyperbolas meet at the other focus.

3.4 Isogonal cubics with respect to $\mathcal{T}$

Let P be a fixed point. The locus of M such that P, M and its isogonal conjugate $M^*_{\mathcal{T}}$ are collinear is an isogonal pivotal cubic with respect to $\mathcal{T}$. This cubic is also inscribed in ABC if and only if it contains the infinite points of the altitudes of ABC since these points are the isogonal conjugates of A, B, C with respect to $\mathcal{T}$.

It follows that all such cubics form a pencil of cubics since they contain nine common identified points. This pencil obviously contains the cubic decomposed into the circumcircle and the line at infinity. It also contains K615 which is the unique other cubic invariant under isogonality with respect to $\mathcal{T}$.

K615 contains X_2, X_3, X_4, X_{64}, X_{154}, X_{3424} and must pass through the in/excenters of $\mathcal{T}$, in particular the incenter X_{5373}. See §4 below.

Figure 13 shows this cubic K615 and the Thomson cubic K002. The four in/excenters of $\mathcal{T}$ are the intersections of two (dashed) diagonal rectangular hyperbolas.

The pencil also contains two special other cubics :
– the one passing through X_6 which is a $\mathcal{K}_0$ (without term in xyz),
– the one passing through X_{376} which is a $\mathcal{K}^+$ (a cubic with three concurring asymptotes).

4 The Thomson triangle and the equiareality center X_{5373}

Mowaffaq Hajja and Panagiotis T. Krasopoulos have studied (see [8]) the following (slightly rephrased) problem. Let X be a point lying inside ABC and let $X_aX_bX_c$ be

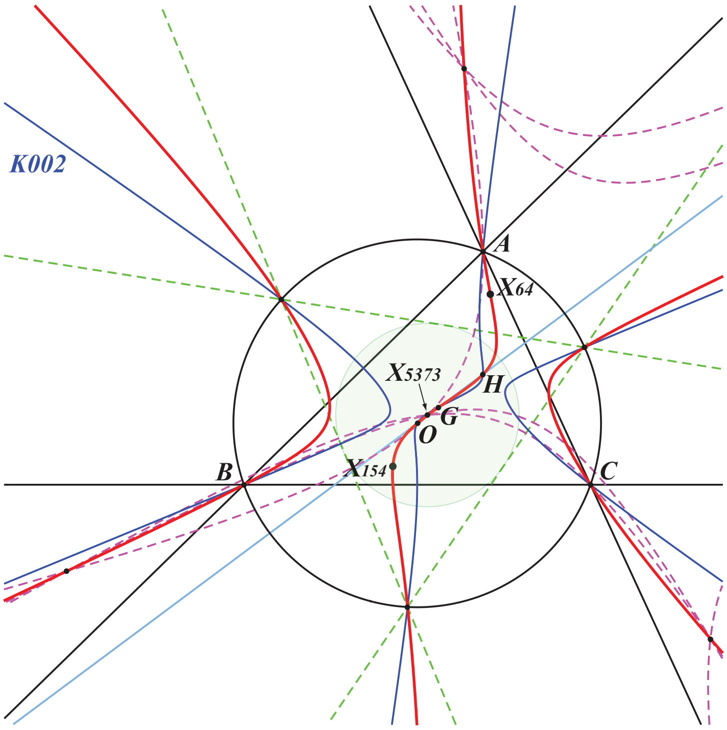

Figure 13: K615 and K002

its pedal triangle. For which X the three quadrilaterals such as AX_cXX_b have the same area, obviously one third of the area of ABC ?

The aforementioned paper supposes that ABC is an acute triangle and that X must be inside ABC. The authors only find one point namely the center X_{5373} in [6] but they do not provide a construction. They do not consider a more general configuration, leaving the problem open.

We will investigate the situation under a different point of view and will show its connexion with the Thomson triangle. In the sequel, all the areas are algebraic and their signs are chosen with respect to the orientation of the reference triangle ABC.

We first take X inside an acute triangle ABC in which case the vertices X_a, X_b, X_c of its pedal triangle lie on the sides of ABC.

In such case, the area of each quadrilateral is the sum of the areas of two rectangular triangles. For example, $[AX_cXX_b] = [AX_cX] + [XX_bA]$ where [...] denotes an area. This rewrites as $[AX_cXX_b] = [AX_cX] - [AX_bX]$ for a better symmetry in the notations.

Let $\alpha(X) = [AX_cXX_b]$, $\beta(X)$ and $\gamma(X)$ being defined likewise. Let Δ be the area of ABC. See figure 14.

After some easy computations we obtain the following propositions.

Proposition 8 *$\alpha(X) = \Delta/3$ if and only if X lies on a rectangular hyperbola denoted h_A.*

h_A has its center at A. Two other rectangular hyperbolas h_B, h_C are defined likewise. These three hyperbolas belong to a same pencil and have four common points forming an orthocentric system. See figure 15.

Proposition 9 *$\beta(X) = \gamma(X)$ if and only if X lies on a rectangular hyperbola denoted H_A.*

H_A has its center at the $A-$vertex of the circumcevian triangle of the Lemoine point K. Two other rectangular hyperbolas H_B, H_C are defined likewise. These three hyperbolas belong to the same pencil as the one mentioned above. See figure 16.

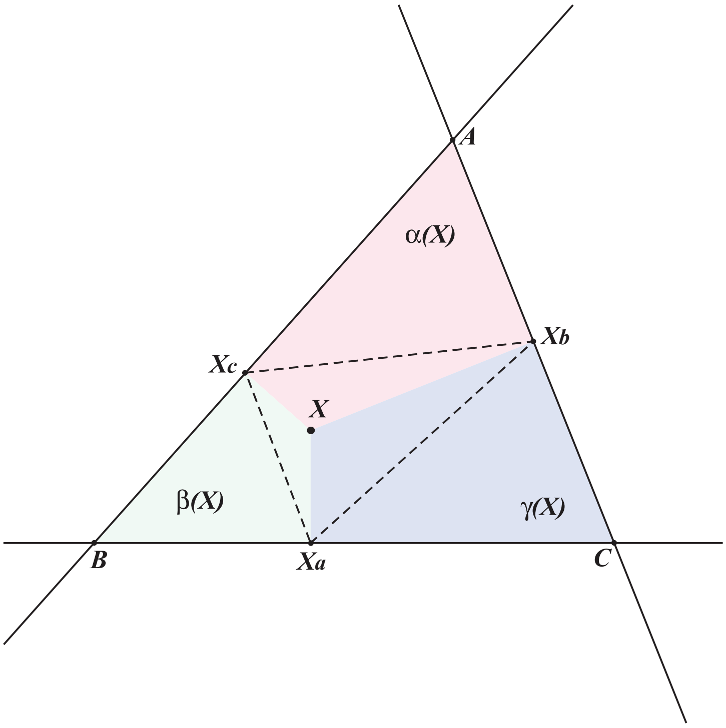

Figure 14: Three quadrilaterals

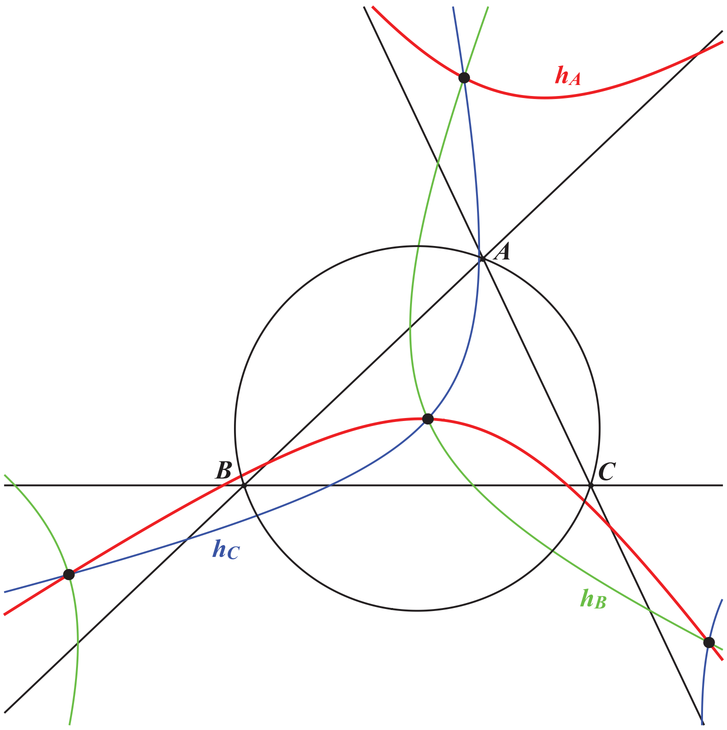

Figure 15: Three hyperbolas h_A, h_B, h_C

It is easy to verify that

Proposition 10 *The six rectangular hyperbolas h_A, h_B, h_C, H_A, H_B, H_C are members of the pencil of diagonal rectangular hyperbolas with respect to $\mathcal{T}$.*

Hence we have

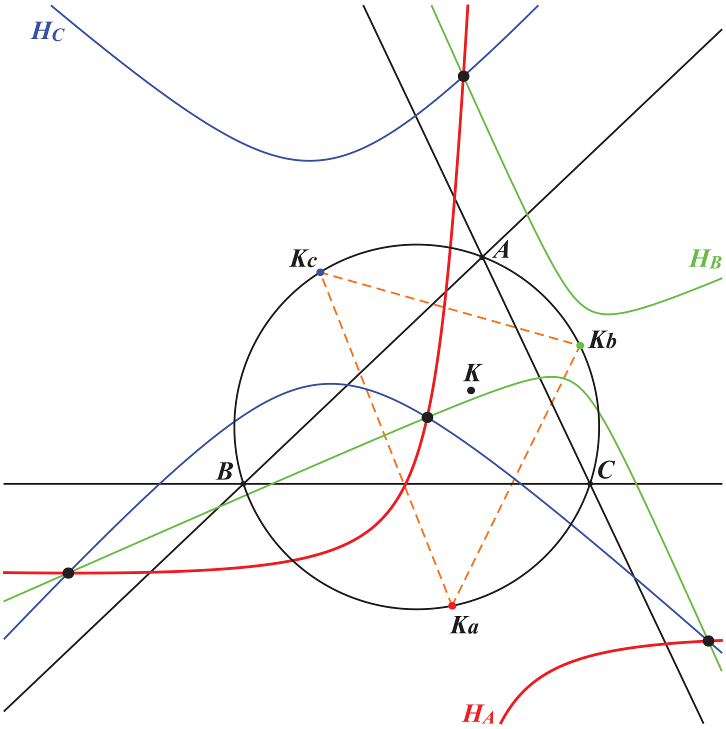

Figure 16: Three hyperbolas H_A, H_B, H_C

Proposition 11 *$\alpha(X) = \beta(X) = \gamma(X)$ if and only if X is one of the four in/excenters of the Thomson triangle $\mathcal{T}$.*

and finally

Proposition 12 *X_{5373} in the incenter of the Thomson triangle $\mathcal{T}$.*

Figure 17 shows X_{5373} on K615 in an acute triangle.

Remark : Recall that X_{5373} and the excenters of $\mathcal{T}$ lie on the diagonal rectangular hyperbolas we met in §1.6 which gives a conic construction of these points.

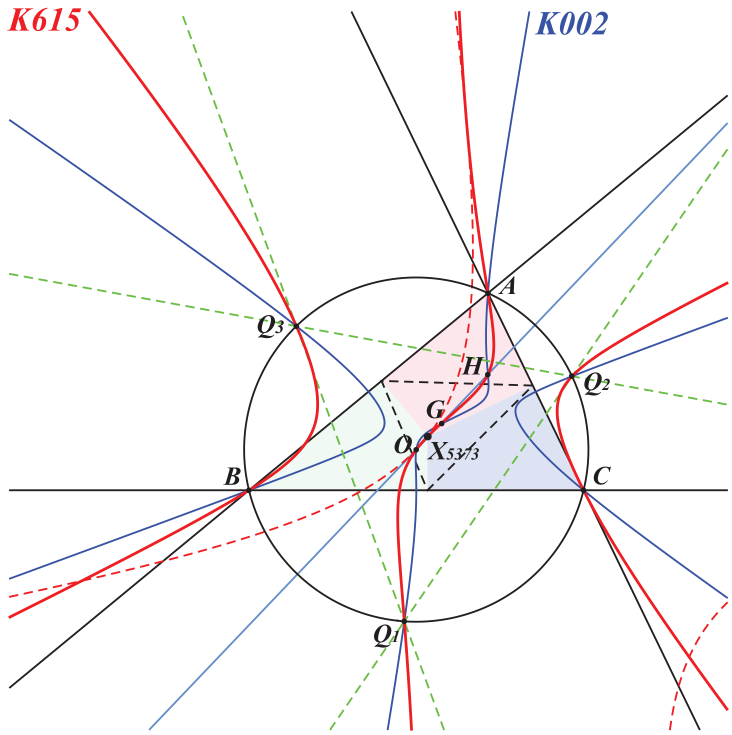

Figure 17: X_{5373}, K615 and K002

5 Appendix : tables of cubics

In this first table, we gather together all the circum-cubics passing through the vertices of $\mathcal{T}$ we have met throughout the paper with some additional interesting examples. X denotes the intersection of concurring asymptotes when the cubic is a $\mathcal{K}^+$.

cubic	type	centers on the cubic	remarks
K002	$p\mathcal{K}$	X_1, X_2, X_3, X_4, X_6, X_9, X_{57}, X_{223}, X_{282}, X_{1073}, X_{1249}, etc	
K167	$p\mathcal{K}$	X_3, X_6, X_{3167}, X_{8770}	
K172	$p\mathcal{K}$	X_3, X_6, X_{25}, X_{55}, X_{56}, X_{64}, X_{154}, X_{198}, X_{1033}, X_{1035}, X_{1436}, X_{7037}	
K280	nodal $sp\mathcal{K}$	X_2, X_6, X_{262}, X_{378}, X_{995}, X_{1002}, X_{1340}, X_{1341}, X_{5968}, X_{7757}	node at G
K297	nodal	X_3, X_6, X_{183}, X_{956}, X_{1344}, X_{1345} , X_{3445}, X_{5968}	node at K
K581	stelloidal $sp\mathcal{K}$	X_2, X_3, X_4, X_{262}	$X = X_{5055}$
K615	$sp\mathcal{K}$, $p\mathcal{K}$ in $\mathcal{T}$	X_2, X_3, X_4, X_{64}, X_{154}, X_{3424}, X_{5373}	
K624	$n\mathcal{K}_0^+$	X_6, X_{523}, X_{2574}, X_{2575} , X_{5968}, X_{8905}, X_{8106}	$X = G$
K625	$n\mathcal{K}_0$	X_6, X_{187}, X_{511}, X_{523}, X_{690}, X_{6137}, X_{6138}	
K626	nodal	X_3, X_{25}, X_{1073}, X_{1384}, X_{1617}, X_{3167}, X_{3420}, X_{3426}	node at O
K759	$sp\mathcal{K}$	X_2, X_3, X_4, X_{3431}, X_{7607}	

The second table shows several non circum-cubics passing through the vertices of $\mathcal{T}$.
Notes :
(1) : K078 is the McCay cubic K003 of $\mathcal{T}$.
(2) : K463 is the focal cubic K187 of $\mathcal{T}$.
(3) : K758 is the isogonal transform of the Thomson cubic K002 with respect to $\mathcal{T}$.

cubic	type	centers on the cubic / notes	remarks
K078	stelloid	X_1, X_2, X_3, X_{165}, X_{5373}, X_{6194} / (1)	$X = X_{3524}$
K138	equilateral	X_2, X_6, X_{5652}	
K463	focal	X_2, X_3, X_{15}, X_{16}, X_{30}, X_{110},X_{5463}, X_{5464} / (2)	focus X_{110}
K609		X_1, X_2, X_3, X_{20},	
K703	$n\mathcal{K}$ in $\mathcal{T}$	?	
K727		X_2, X_3, X_{7712}	
K758	central	X_2, X_3, X_{154}, X_{165}, X_{376}, X_{3576} / (3)	$X = X_3$

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