Perspective Triangles inscribed in the Neuberg Cubic

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Abstract

We show that the Neuberg cubic $\mathcal{N}$ = K001 is a pivotal cubic in infinitely many triangles and we explicitely construct these triangles. This is a generalization of the 25 triangles of Table 19 in [2].

1 Introduction

The Neuberg cubic $\mathcal{N}$ is one of the most documented cubics in triangle geometry. Basically it is an isogonal pivotal cubic, the locus of M such that M and its isogonal conjugate M^* lie on a parallel to the Euler line whose point at infinity is X_{30}. Hence M, M^* and the pivot X_{30} are collinear.

The isopivot $X_{74} = X_{30}^*$ is the tangential of X_{30} and, for any M on $\mathcal{N}$, the points X_{74}, M and $M' = X_{30}/M$ are also collinear. M' is the $X_{30}-$Ceva-conjugate of M.

This cubic has many interesting properties which are not mentioned here, see [2] for further informations.

In this paper, we shall focus on pairs of perspective triangles inscribed in $\mathcal{N}$ such as the 25 triangles presented in Table 19 of [2].

Throughout this paper, M, N are two distinct points on $\mathcal{N}$ with isogonal conjugates M^*, N^* and $X_{30}-$Ceva-conjugates M', N' respectively. The tangential of M is denoted $\widetilde{M}$.

The cevian triangle of X_{30} is denoted $AoBoCo$. In other words, Ao is the intersection of BC and the parallel at A to the Euler line. $AoBoCo$ is obviously inscribed in $\mathcal{N}$ and the two triangles ABC and $AoBoCo$ are perspective at X_{30}.

2 Constructions related with one point on $\mathcal{N}$.

Like many other proofs in this paper, the 3x3 table shows sets of collinear points on $\mathcal{N}$ lying on six lines represented by the three lines and the three columns of the table. If five of these collinearities are verified then the sixth must also be verified. This is a classical property of cubics.

The additional bottom line and right-hand column both give a justification of these collinearities : $(*)$ indicates that two isogonal conjugates are collinear with X_{30} and $(/)$ indicates that two $X_{30}-$Ceva-conjugates are collinear with X_{74}. The black triangle symbol indicates the proven collinearity.

2.1 Third points Ma on AM^* and Ma^* on AM

Theorem 1. *$Ma = AM^* \cap AoM'$ and $Ma^* = AM \cap MaX_{30}$ lie on $\mathcal{N}$.*

Let Ma be the third point of $\mathcal{N}$ on the line AM^*.

A	M^*	Ma	definition of Ma
X_{74}	M	M'	$(/)$
A	X_{30}	Ao	$(*)$
$(/)$	$(*)$	$\blacktriangle$	

The line AM meets $\mathcal{N}$ at Ma^* since the isogonal transform of the line AM^*Ma is the line AM.

The points Mb and Mc are defined similarly. Obviously, the triangles $MaMbMc$ and ABC (resp. $AoBoCo$) are perspective at M^* (resp. M') and the triangles $Ma^*Mb^*Mc^*$ and ABC (resp. $AoBoCo$) are perspective at M (resp. $(M^*)'$).

2.2 Tangentials

Theorem 2. *The points M, Ma, Mb, Mc share the same tangential $\widetilde{M}$.*

M	M	$\widetilde{M}$	definition of $\widetilde{M}$
M^*	A	Ma	Theorem 1
X_{30}	Ma^*	Ma	$(*)$
$(*)$	Theorem 1	$\blacktriangle$	

The table shows that $\widetilde{M} = \widetilde{Ma}$, the other points are treated likewise.

It follows that the polar conic $P(\widetilde{M})$ of $\widetilde{M}$ in $\mathcal{N}$ passes through M, Ma, Mb, Mc and obviously $\widetilde{M}$.

Construction of $\widetilde{M}$

The tangents at M, M^* to $\mathcal{N}$ meet at E^* where E is the midpoint of M, M^*. This latter point lies on the polar conic of X_{30} in $\mathcal{N}$.

Hence, following [1], §1.4.3, we obtain

$$\widetilde{M} = ME^* \cap (M^*)'(M')^* \quad \text{and} \quad \widetilde{M^*} = M^*E^* \cap M'M^{*\prime *}.$$

Since the tangentials of the three collinear points X_{30}, M, M^* must also be collinear, we obtain that X_{74}, $\widetilde{M}$, $\widetilde{M^*}$ are collinear and then $\widetilde{M}$, $\widetilde{M^*}$ are two $X_{30}-$Ceva-conjugate points on $\mathcal{N}$ that is $\widetilde{M^*} = (\widetilde{M})'$.

Theorem 3. *The points M', M^*, $(\widetilde{M})^*$ are collinear.*

M'	M^*	$(\widetilde{M})^*$	$\blacktriangleleft$
M	M	$\widetilde{M}$	definition of $\widetilde{M}$
X_{74}	X_{30}	X_{30}	$(*)$
$(/)$	$(*)$	$(*)$	

2.3 Other points on $\mathcal{N}$ and consequences

Let $M'a$, $M'b$, $M'c$ be the third points of $\mathcal{N}$ on the sidelines of triangle $MaMbMc$ and $M'a^*$, $M'b^*$, $M'c^*$ their isogonal conjugates.

Theorem 4. *(1) The points Ma, C, Mb^* are collinear.*
(2) The points M, Ma, $M'a$ are collinear.
(3) The points A, $\widetilde{M}$, $M'a^*$ are collinear.
(4) The lines $M(\widetilde{M})^*$, $MaM'a^*$, $MbM'b^*$, $McM'c^*$ concur at S which lies on $\mathcal{N}$.

Proof (1) :

X_{30}	Mb	Mb^*	$(*)$
Ao	B	C	definition of Ao
A	M^*	Ma	definition of Ma
$(*)$	definition of Mb	▲	

Five other collinearities are obtained likewise such as A, Mb, Mc^* we shall use in the next proof.

Proof (2) :

Mb	Mc	$M'a$	definition of $M'a$
A	M^*	Ma	definition of Ma
Mc^*	C	M	Theorem 1
Proof (1)	definition of Mc	▲	

Similarly, the points M, Mb, $M'b$ and M, Mc, $M'c$ are collinear.

Proof (3) :

A	$\widetilde{M}$	$M'a^*$	◄
Ma	M	$M'a$	Proof (2)
M^*	M	X_{30}	$(*)$
definition of Ma	definition of $\widetilde{M}$	$(*)$	

Proof (4) :
Let S be the third point of $\mathcal{N}$ on the line $M(\widetilde{M})^*$.

S	M	$(\widetilde{M})^*$	definition of S
X_{30}	M	M^*	$(*)$
S^*	$\widetilde{M}$	M'	◄
$(*)$	definition of $\widetilde{M}$	Theorem 3	

This shows that S^*, $\widetilde{M}$ and M' are collinear.

$M'a^*$	Ma	S	◄
$\widetilde{M}$	M'	S^*	see above
A	Ao	X_{30}	$(*)$
Proof (3)	definition of Ma	$(*)$	

Hence S lies on the line $MaM'a^*$ and similarly on the lines $MbM'b^*$, $McM'c^*$. In other words, the triangles $MaMbMc$ and $M'a^*M'b*M'c^*$ are perspective at S.

2.4 The Neuberg cubic in triangle $MaMbMc$

We have seen that, for any point M on $\mathcal{N}$, the tangents at Ma, Mb, Mc concur at $\widetilde{M}$ and that $\mathcal{N}$ meets the sidelines of $MaMbMc$ at the feet of the cevian lines of M in $MaMbMc$. Since M and $\widetilde{M}$ obviously lie on $\mathcal{N}$, we obtain

Theorem 5. *The Neuberg cubic is a pivotal cubic with pivot M and isopivot $\widetilde{M}$ in any triangle $MaMbMc$.*

Let $\mathcal{C}(M)$ be the circumcircle of $MaMbMc$. Since $\mathcal{N}$ is a circular cubic, it meets $\mathcal{C}(M)$ at the two circular points at infinity J_1 and J_2, the vertices of $MaMbMc$ and a sixth point which turns out to be S.

Theorem 6. *The circumcircle $\mathcal{C}(M)$ of $MaMbMc$ passes through S.*

Proof : let $\mathcal{F}a$ be the pencil of circles passing through Ma and S. Each circle meets $\mathcal{N}$ again at two (finite) points. The line through these two points passes through a fixed point F of $\mathcal{N}$ which is the coresidual of Ma, S, J_1 and J_2.

The union of the line at infinity and the line SMa is a degenerate circle of $\mathcal{F}a$ meeting $\mathcal{N}$ at X_{30} and $M'a^*$ (see Theorem 4, (4)). It follows that F must be the third point of $\mathcal{N}$ on the line $X_{30}M'a^*$ hence $F = M'a$.

Consider now the circle of $\mathcal{F}a$ which contains Mb. This must meet $\mathcal{N}$ again at a point on the line $MbM'a$ hence this point must be Mc. Consequently, the points S, Ma, Mb, Mc are concyclic and this completes the proof.

2.5 Example

Let us take $M = X_4$ hence $M^* = X_3$, $M' = X_{5667}$, $(M')^* = X_{8431}$, $\widetilde{M} = X_{3484}$, $(\widetilde{M})^* = X_{8439}$, $\widetilde{M^*} = X_{1157}$ and finally $S = X_{5684}$. S^* is unlisted in ETC.

Ma, Mb, Mc are the third points on the cevian lines of X_3 and this gives triangle #4 in Table 19. See figure 1.

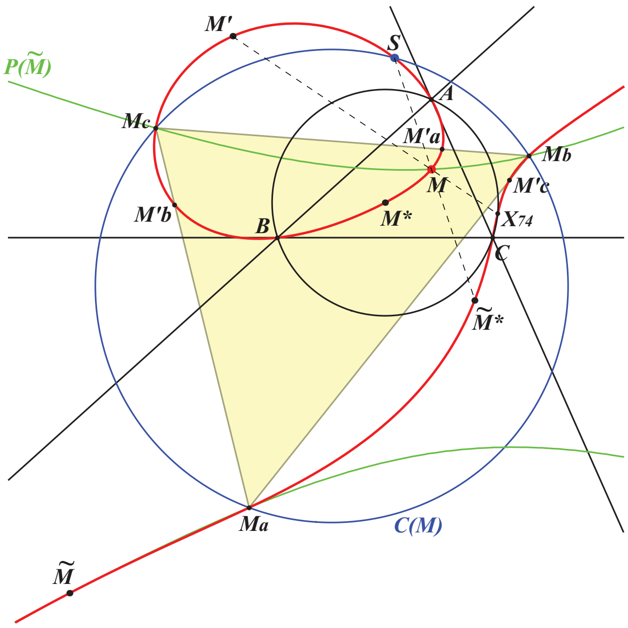

Figure 1: A triangle $MaMbMc$ inscribed in $\mathcal{N}$

3 Constructions related with two points on $\mathcal{N}$.

3.1 Theorems

Let M, N be two distinct points on $\mathcal{N}$.

Theorem 7. *The line MN meets $\mathcal{N}$ again at X such that $X^* = M'N^* \cap M^*N'$.*

Proof : Let X^* be the third point $\mathcal{N}$ on the line $M'N^*$.

M'	N^*	X^*	definition of X^*
Ma	A	M^*	definition of Ma
Ao	Na	N'	Theorem 1
Theorem 1	definition of Na	$\blacktriangle$	

Hence X^* is the third point $\mathcal{N}$ on the line M^*N'.
It is easy to check now that the line MN contains X.

X	X^*	X_{30}	$(*)$
M	M'	X_{74}	definition of M'
N	N^*	X_{30}	$(*)$
$\blacktriangle$	see above	$(/)$	

Theorem 8. *The triangles $MaMbMc$ and $NaNbNc$ are inscribed in $\mathcal{N}$ and perspective at X.*

Proof : we show that the points X, Ma, Na are collinear.

Ma	Na	X	$\blacktriangleleft$
M^*	N'	X^*	Theorem 7
A	Ao	X_{30}	$(*)$
definition of Ma	Theorem 1 for N	$(*)$	

Similarly, the points X, Mb, Nb and X, Mc, Nc are collinear.

3.2 Example

Let us take $M = X_{13}$ and $N = X_{14}$ hence $X = X_{399}$.

We have : $M^* = X_{15}$, $M' = X_{5623}$, $\widetilde{M} = X_{5674}$, $(\widetilde{M})^* = X_{8491}$, $\widetilde{M^*} = X_{1337}$,

and : $N^* = X_{16}$, $N' = X_{5624}$, $\widetilde{N} = X_{5675}$, $(\widetilde{N})^* = X_{8492}$, $\widetilde{N^*} = X_{1338}$.

Ma, Mb, Mc are the third points on the cevian lines of X_{15} and this gives triangle #7 in Table 19. Similarly, Na, Nb, Nc are the third points on the cevian lines of X_{16} and this gives triangle #8 in Table 19. This explains why these two triangles are perspective at X_{399}. See figure 2.

Obviously, $\mathcal{N}$ is a pivotal cubic in both triangles $MaMbMc$, $NaNbNc$ with pivots M, N and isopivots $\widetilde{M}$, $\widetilde{N}$ respectively.

3.3 Conclusion

If X is a given point on $\mathcal{N}$, any line through X meets $\mathcal{N}$ again at two points M, N associated with two perspective triangles $MaMbMc$, $NaNbNc$.

For instance, taking $X = X_{399}$ as above, it is easy to find several simple lines passing through X which explain the existence of the six pairs of perspective triangles in Table 19. Naturally, one can find many other lines but the coordinates of the corresponding triangle centers are often very complicated.

With $X = X_{30}$, twelve pairs of perspective triangles are clearly identified.

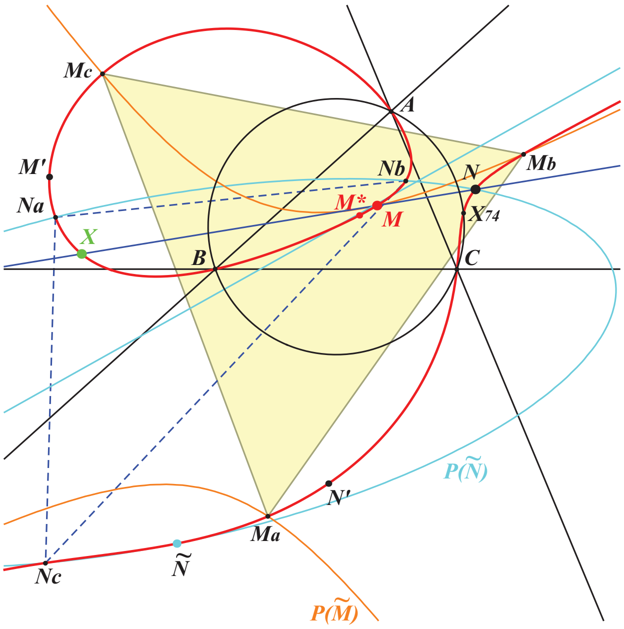

Figure 2: Two perspective triangles $MaMbMc$ and $NaNbNc$ inscribed in $\mathcal{N}$

4 Connection with prehessians

Like any other pivotal cubic, $\mathcal{N}$ has three real prehessians $\mathcal{N}_a$, $\mathcal{N}_b$, $\mathcal{N}_c$.

Each prehessian $\mathcal{N}_i$ is associated with a conjugation F_i which maps any point M on $\mathcal{N}$ to the center of its (degenerate) polar conic in $\mathcal{N}_i$. $F_a(M)$ is the point Ma as above, the other points likewise.

References

[1] J.P. Ehrmann and B. Gibert, *Special Isocubics in the Triangle Plane*, available at http://bernard.gibert.pagesperso-orange.fr

[2] B. Gibert, *Cubics in the Triangle Plane*, available at http://bernard.gibert.pagesperso-orange.fr

[3] C. Kimberling, *Triangle Centers and Central Triangles*, Congressus Numerantium 129, pp. 1-295, 1998.

[4] C. Kimberling, *Encyclopedia of Triangle Centers*, 2000-2015 http://faculty.evansville.edu/ck6/encyclopedia/ETC.html