

The Lemoine Pencils of Cubics

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Abstract : This note is a complement and another generalization of our previous paper *The Lemoine Cubic and its Generalizations*. See [3].

1 The Lemoine pencil

1.1 The general case

Let P be a point in the plane of reference triangle ABC . The cevian lines of P meet the perpendicular bisectors of ABC at A' , B' , C' . In [3], we recalled that the points A' , B' , C' are collinear if and only if P lies on the Lemoine cubic $\mathcal{K}(O) = \mathbf{K009}$. This locus can also be seen as the locus of point P such as the area of triangle $A'B'C'$ equals 0.

Now, for any real number k , we consider the locus $\mathcal{K}(O; k)$ of P such that the (algebraic) area of triangle $A'B'C'$ is constant and equal to $k\Delta$, where Δ is the area of ABC . An easy computation gives

Theorem 1 $\mathcal{K}(O; k)$ is a cubic passing through A, B, C, H , the projections O_a, O_b, O_c of O on the altitudes of ABC .

All these cubics belong to the same pencil of cubics $\mathcal{F}_O$ we call the *Lemoine pencil* which contains the Lemoine cubic itself (obtained for $k = 0$) and the degenerated cubic into the union of the altitudes (obtained for $k = \infty$). Hence, the tangent at H and the polar conic of H are independent of k . It follows that all these cubics have the same osculating circle Γ_H at H . More precisely,

- the tangent $\mathcal{T}_H$ at H passes through X_{125} (the center of the Jerabek hyperbola) and X_{74} (the isogonal conjugate of the infinite point of the Euler line).
- the polar conic $\mathcal{C}_H$ of H is a rectangular hyperbola through O, H . It follows that $\mathcal{K}(O; k)$ meets the Euler line at H and two other points harmonically conjugated with respect to H and O .

More generally, the polar conic of any point on the Euler line is a rectangular hyperbola and, when $k = -1$, the polar conic of any point of the plane is a rectangular hyperbola. In this case, the cubic is said to be an equilateral cubic. See $\mathcal{K}(O; -1)$ below.

1.2 Some special cases

Apart the Lemoine cubic $\mathcal{K}(O) = \mathcal{K}(O; 0) = \mathbf{K009}$, a nodal cubic which is detailed in [3], and the degenerate $\mathcal{K}^+$ which is the union of the altitudes, $\mathcal{F}_O$ contains several other interesting cubics :

- One circular cubic which is $\mathcal{K}(O; 1)$ passing through $X_4, X_5, X_{30}, X_{1141}$. Its real asymptote is the parallel at X_{110} to the Euler line which is the orthic line (or axis) of the cubic. The singular focus (not on the cubic) is the common point of the lines $X_2X_{399}, X_3X_{2888}, X_5X_{125}, X_{30}X_{74}$, etc. See figure 1.

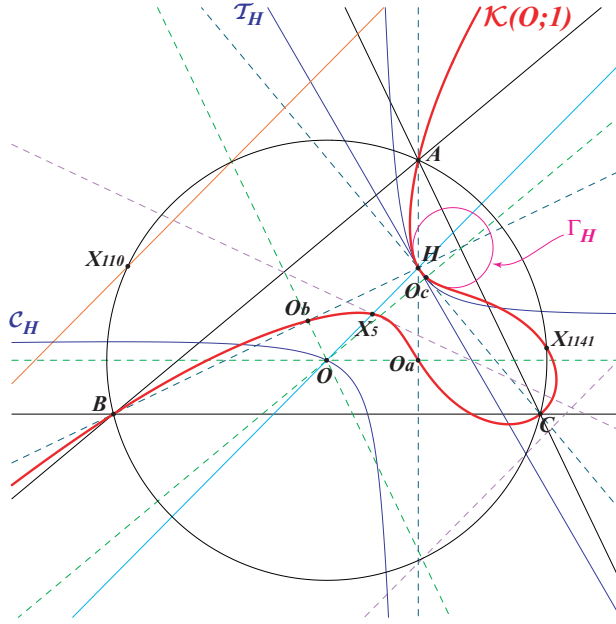


Figure 1: The circular cubic $\mathcal{K}(O;1)$

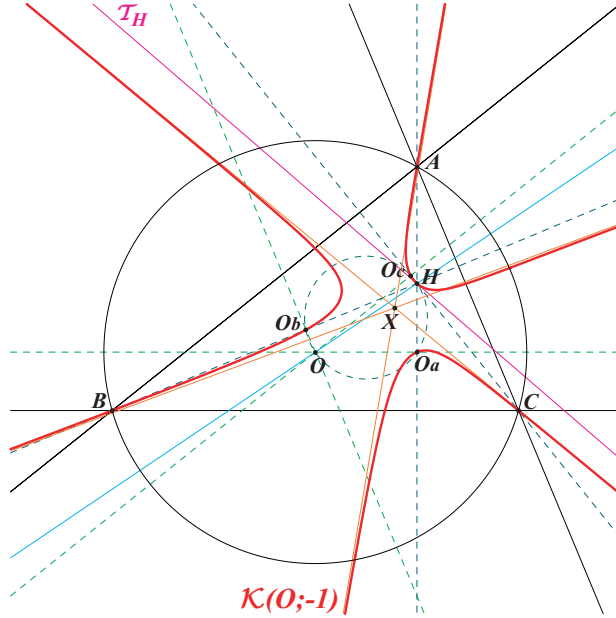


Figure 2: The equilateral cubic $\mathcal{K}(O;-1)$

- One $\mathcal{K}_{60}^+$ which is $\mathcal{K}(O;-1)$ having three real asymptotes parallel to those of the McCay cubic concurring at the intersection of the lines X_2X_{568} , X_3X_{143} , X_5X_{389} , X_6X_{1511} , $X_{30}X_{51}$, etc. See figure 2.
- A third $\mathcal{K}^+$ which is $\mathcal{K}(O;1/5)$ having three concurring asymptotes at X_{549} , the midpoint of GO . Its asymptotes are parallel to those of $p\mathcal{K}(X_6, P)$ where P is defined by $\overrightarrow{OP} = 3/2 \overrightarrow{OH}$. It meets the circumcircle at the same points as $p\mathcal{K}(X_6, P')$ where P' is defined by $\overrightarrow{OP'} = 3/4 \overrightarrow{OX_{2979}}$. See figure 3.
- Three non-pivotal isocubics, one of them $n\mathcal{K}_1$ when $k = \frac{1}{8 \cos A \cos B \cos C}$.

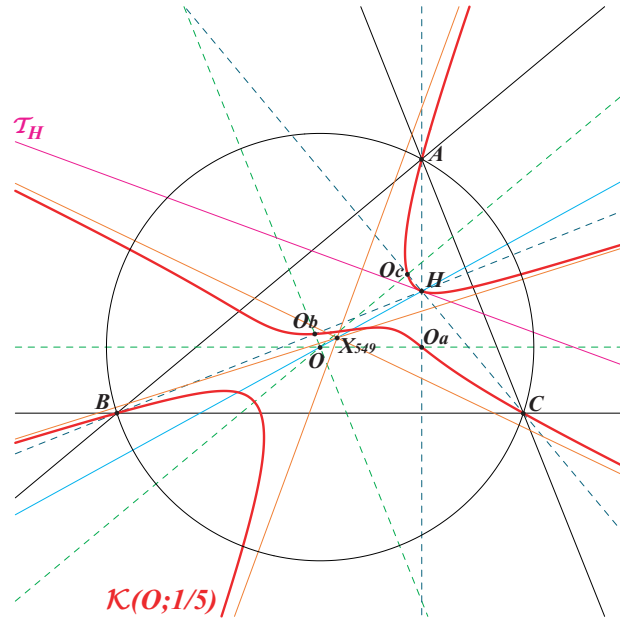


Figure 3: The cubic $\mathcal{K}(O;1/5)$

This passes through H , X_{54} , X_{74} . Its pole and its root are the isogonal conjugates of X_{343} and X_{1625} respectively.

The two other cubics $n\mathcal{K}_2$ and $n\mathcal{K}_3$ are very complicated. See figure 4.

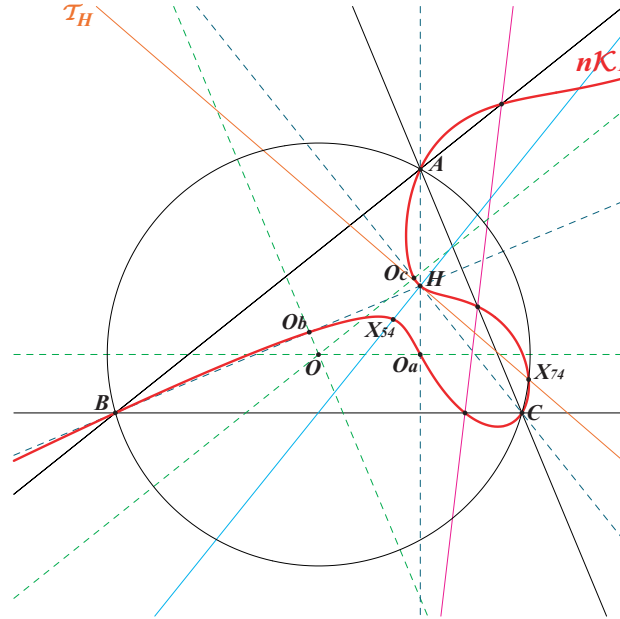


Figure 4: The cubic $n\mathcal{K}_1$

2 The generalized Lemoine pencils

2.1 The general case

All what is said in the first section can be generalized when we replace O by any fixed finite point $Q \neq H$ ¹ and the perpendicular bisectors of ABC by the perpendiculars at Q to the sidelines of ABC . The Lemoine cubic $\mathcal{K}(O)$ is then replaced by the generalized Lemoine cubic $\mathcal{K}(Q)$ as seen in [3]. We obtain a generalized version of theorem 1 :

Theorem 2 *The locus $\mathcal{K}(Q; k)$ of P such as the (algebraic) area of triangle $A'B'C'$ is constant and equal to $k\Delta$ is a cubic passing through A, B, C, H , the projections Q_a, Q_b, Q_c of Q on the altitudes of ABC .*

For a given point Q , all these cubics $\mathcal{K}(Q; k)$ belong to the same pencil of cubics $\mathcal{F}_Q$ we call the *generalized Lemoine pencil* associated to Q . Thus, if M is a known point on $\mathcal{K}(Q; k)$, the cubic may be denoted by $\mathcal{K}(Q; M)$.

This pencil contains the generalized Lemoine cubic $\mathcal{K}(Q; 0)$ itself which is the only cubic passing through Q and the degenerated cubic $\mathcal{K}(Q; \infty)$ which is the union of the altitudes.

Here again, the tangent $\mathcal{T}_H$ at H to the cubic and the polar conic $\mathcal{C}_H$ of H are independent of k . $\mathcal{T}_H$ is the tangent at H to the rectangular circum-hyperbola which passes through the infinite point of HQ .

$\mathcal{K}(Q; k)$ meets the line HQ at H and two other points Q_1, Q_2 such that

$$\overrightarrow{HQ_1} = \frac{1}{1 + \sqrt{k}} \overrightarrow{HQ} \quad \text{and} \quad \overrightarrow{HQ_2} = \frac{1}{1 - \sqrt{k}} \overrightarrow{HQ}.$$

These points are harmonically conjugated with respect to H and Q . They coincide when $k = 0$, the case of the nodal cubic $\mathcal{K}(Q; 0)$.

2.2 Special cubics of $\mathcal{F}_Q$

For a given point $Q \neq H$, $\mathcal{F}_Q$ contains :

- apart the union of the altitudes, at most two $\mathcal{K}^+$ i.e. cubics with concurring asymptotes. Their equations are generally complicated unless Q lies on the Euler line. See below.
- one circular cubic which is $\mathcal{K}(Q; 1)$ passing through the midpoint of HQ and the infinite point of the line HQ .

This circular cubic is a focal cubic if and only if the complement of Q lies on the McCay cubic **K005**.

- one equilateral $\mathcal{K}_{60}$ cubic if and only if Q lies on the Euler line and this cubic is always a $\mathcal{K}_{60}^+$. Its asymptotes are parallel to those of the McCay cubic and concur at a point on the line passing through X_{381} (midpoint of HG) parallel to the line OX_{110} .

Its six common points with the circumcircle lie on an isogonal $p\mathcal{K}$ whose pivot is a point of the cubic $\mathcal{K}_{60}^+$.

All these $\mathcal{K}_{60}^+$ form a pencil of cubics and this pencil contains at least one, at most three $\mathcal{K}_{60}^{++}$.

Since we know one of the two $\mathcal{K}^+$ of $\mathcal{F}_Q$, the other one is easy to determinate. Its asymptotes concur on the Euler line at the midpoint of GQ .

¹When $Q = H$, the triangles $A'B'C'$ and ABC coincide. If $k = 1$, the locus of P is the whole plane and if $k \neq 1$, the locus of P is empty.

If Q is taken on the Kiepert hyperbola, $\mathcal{K}(Q; k)$ becomes a $\mathcal{K}_0$ i.e. a cubic without term in xyz . In this case, $\mathcal{F}_Q$ always contains a pivotal cubic $p\mathcal{K}$ with pole on the circum-hyperbola passing through G and K .

2.3 The pencil $\mathcal{F}_G$

The most interesting case is probably $Q = G$ (on the Kiepert hyperbola and on the Euler line). These cubics pass through A, B, C, H , the projections of G on the altitudes, these three latter points on the orthocentroidal circle.

The tangent at H is the line through $X_{74}, X_{107}, X_{125}, X_{133}$, etc. See figures 5 and 6.

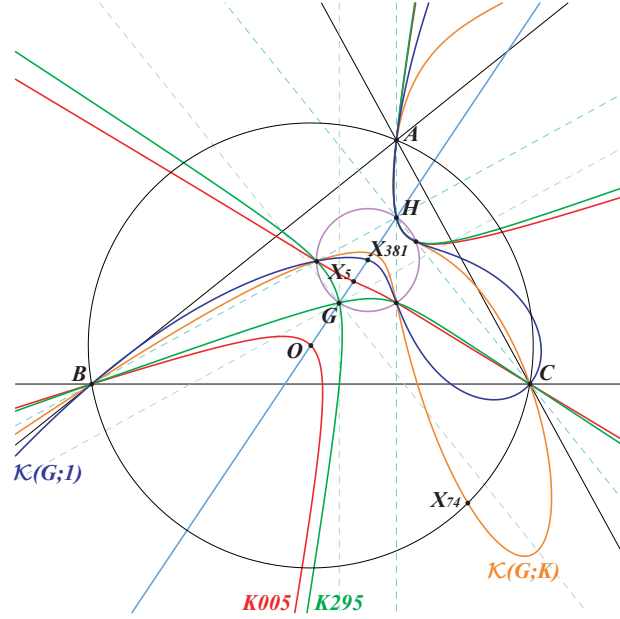


Figure 5: Cubics of $\mathcal{F}_G$ (part 1)

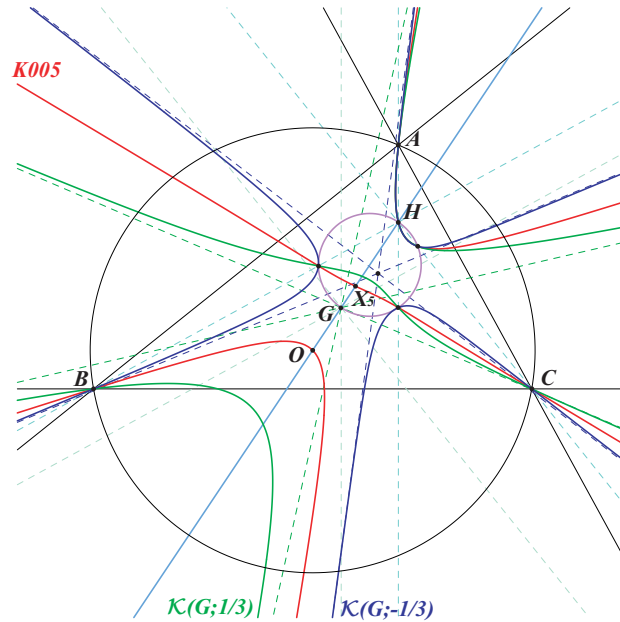


Figure 6: Cubics of $\mathcal{F}_G$ (part 2)

This pencil $\mathcal{F}_G$ contains :

- one $p\mathcal{K}$, the Napoleon cubic **K005** which is $\mathcal{K}(G; 1/9)$. This gives another characterization of **K005** : locus of P such that the cevian lines of P meet the pedal lines of G at three points A', B', C' verifying $\text{area}(ABC) = 9 \text{ area}(A'B'C')$. These two triangles are orthologic at G and another point, collinear with G and P , lying on a quintic.
- one nodal cubic, **K295** which is $\mathcal{K}(G; 0)$.
- one circular cubic passing through $X_4, X_{13}, X_{14}, X_{30}, X_{381}$ which is $\mathcal{K}(G; 1)$.
- one $\mathcal{K}_{60}^+ = \mathcal{K}(G; -1/3)$ passing through X_4, X_{568} having three asymptotes concurring at X such that $\overrightarrow{GX} = 2/3 \overrightarrow{GX_{51}}$ or equivalently $\overrightarrow{X_5X} = 1/3 \overrightarrow{X_5X_{568}}$.
- one $\mathcal{K}^+ = \mathcal{K}(G; 1/3)$ passing through $X_4, X_{485}, X_{486}, X_{2043}, X_{2044}$ having three asymptotes concurring at G and parallel to those of the Orthocubic **K006**.
- three $n\mathcal{K}_0$. One of them is $n\mathcal{K}_0(X_{25}, X_{523}, X_4)$ passing through $X_4, X_6, X_{74}, X_{1990}$. The two other are not always real and complicated.

2.4 The pencil $\mathcal{F}_{Na}$

Now, we take $Q = Na$ (Nagel point) whose complement is the incenter of ABC , a point on the McCay cubic. It follows that $\mathcal{F}_{Na}$ contains the focal cubic $\mathcal{K}(Na; 1)$ with singular focus X_{80} passing through $X_1, X_4, X_{80}, X_{355}, X_{517}$. See figure 7.

$\mathcal{K}(Na; 1)$ also contains :

- the projections of Na on the altitudes, these points lying on the Fuhrmann circle with diameter HN_a and center X_{355} ,
- the intersection E of the lines X_1X_3, X_4X_{80} with first barycentric coordinate $a(2b S_B + 2c S_C - abc)$.
- two (not always real) points E_1, E_2 on the orthic line (or axis) X_5X_{10} of the cubic which can therefore be seen as the locus of contacts of tangents drawn from X_{80} to the circles passing through these two points.

Naturally, $\mathcal{F}_{Na}$ also contains the generalized Lemoine cubic $\mathcal{K}(Na; 0)$ passing through $X_4, X_8, X_{145}, X_{280}$.

$\mathcal{K}(Na; 4)$ is the cubic of the pencil which passes through the Gergonne point X_7 and also X_{962} .

2.5 The pencil $\mathcal{F}_L$

When $Q = L = X_{20}$ (de Longchamps point), we obtain another pencil containing a focal cubic. This is the Euler isogonal focal cubic **K187** in [2]. This has singular focus X_{74} and passes through X_3, X_4, X_{30}, X_{74} and the foci of the inscribed Steiner ellipse. **K187** is the locus of foci of inscribed conics centered on the Euler line. It is an isogonal focal $n\mathcal{K}$ with root X_{525} . The real asymptote is parallel to the Euler line at X_{110} .

The pencil contains **K041** = $\mathcal{K}(L; 0)$ and the equilateral cubic $\mathcal{K}(L; -3)$. See figure 8.

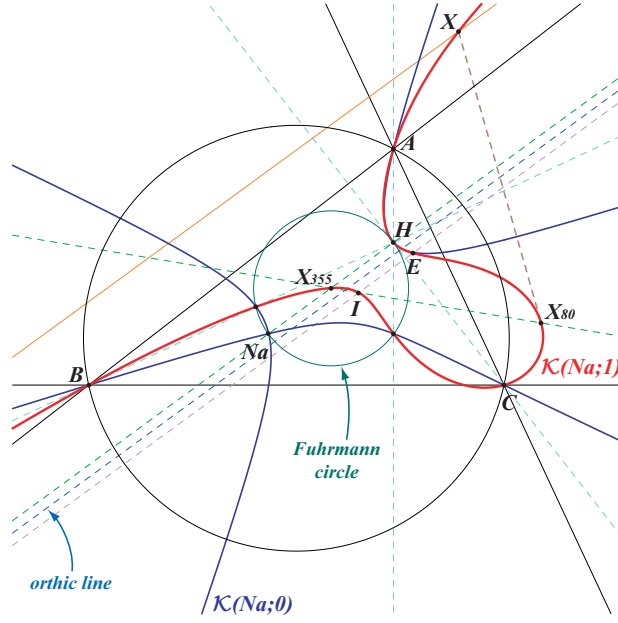


Figure 7: Cubics of $\mathcal{F}_{Na}$

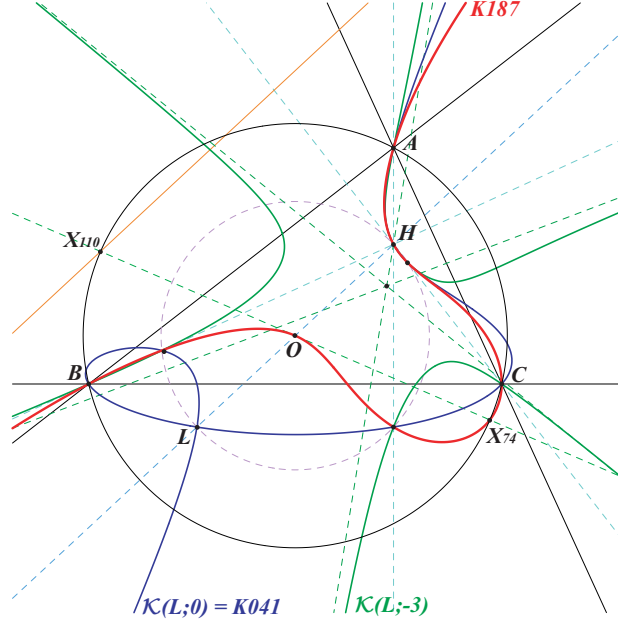


Figure 8: Cubics of $\mathcal{F}_L$

2.6 The pencil $\mathcal{F}_{H'}$

In this paragraph, $H' = X_{376}$ denotes the reflection of H in G , also the reflection of G in O and the midpoint of GL (L is the de Longchamps point X_{20}).

The main interest is that this is the only pencil containing a non degenerate central cubic namely $\mathcal{K}(H'; 1/9)$. This has center O , inflexion point with tangent passing through X_{74} and X_{110} . The cubic contains H and L . See figure 9.

$\mathcal{F}_{H'}$ also contains :

– $\mathcal{K}(H'; -5/3)$, a $\mathcal{K}_{60}^+$ with asymptotes concurring at X on the Brocard

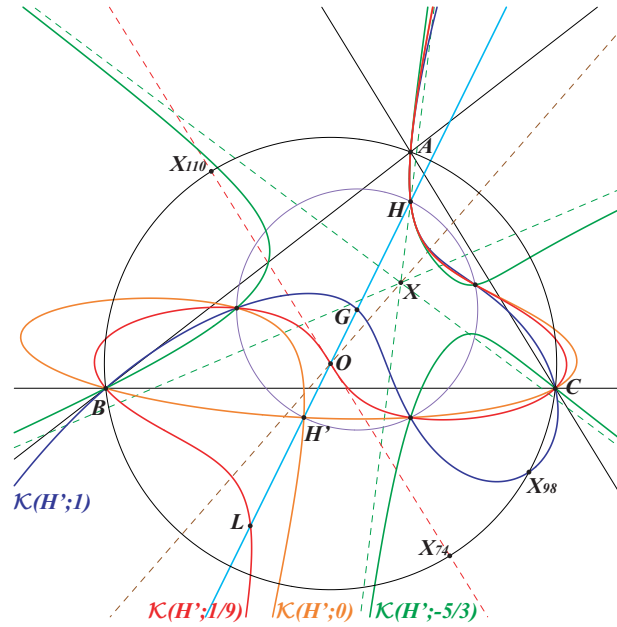


Figure 9: Cubics of $\mathcal{F}_{H'}$

axis such that $\overrightarrow{OX} = 2/3 \overrightarrow{OX_{568}}$,

- $\mathcal{K}(H'; 1)$, a circular cubic passing through G , X_{30} and X_{98} ,
- $\mathcal{K}(H'; 0)$, a Lemoine generalized cubic.

All these cubics meet on the circle with center G and diameter HH' .

References

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