

# Inverses of Isocubics

Bernard Gibert

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## Abstract

We study the isocubics whose inverse (in the circumcircle) is also an isocubic.

## 1 Generalities

Let us consider two points  $\Omega = p : q : r$  (the pole of the isoconjugation) and  $P = u : v : w$  (the pivot or the root of the isocubic).

We briefly recall that an isocubic is a circumcubic invariant by isoconjugation. It is necessarily of one of the two following types (with respect to the reference triangle  $ABC$ ) :

- a pivotal isocubic or  $p\mathcal{K}$ , with pivot  $P$  and equation

$$\sum_{\text{cyclic}} ux(ry^2 - qz^2) = 0,$$

- a non-pivotal isocubic or  $n\mathcal{K}$ , with root  $P$  and equation

$$\sum_{\text{cyclic}} ux(ry^2 + qz^2) + kxyz = 0.$$

An isocubic meets the sidelines of  $ABC$  again at three points  $U, V, W$  which are

- the vertices of the cevian triangle of  $P$  for a  $p\mathcal{K}$ ,
- the traces of the trilinear polar  $\mathcal{L}(P)$  of  $P$  for a  $n\mathcal{K}$ .

See [1] and [2] for more detailed informations.

In general, the inverse of a circumcubic is a circumsextic since the inversion (in the circumcircle of  $ABC$ ) is a quadratic involution. Hence, if we want that the inverse of the cubic be a cubic again, the former (and therefore the latter) must contain the singular points of the inversion namely the circular points at infinity and the circumcenter  $O$ .

It follows that the inverse  $\mathcal{K}'$  of an isocubic  $\mathcal{K}$  is a (circum)cubic if and only if  $\mathcal{K}$  is a circular cubic passing through  $O$ .

This necessary condition is obviously not sufficient. This is examined in the following paragraphs.

## 2 Pivotal isocubics

Let  $\mathcal{K} = p\mathcal{K}(\Omega, P)$  be a pivotal circular cubic passing through  $O$  and let  $\mathcal{K}'$  be its inverse in the circumcircle.

**Theorem 1**  $\mathcal{K}'$  is also a pivotal cubic (with pole  $\Omega'$  and pivot  $P'$ ) if and only if either :

- (1)  $\Omega$  lies on the circumconic  $\mathcal{C}(X_{184})$  with perspector  $X_{184}$  and  $P$  lies on the circumcircle,
- (2)  $\Omega$  lies on the Brocard axis and  $P$  lies on the Euler line.

These two situations are detailed in the two following sections.

Notes on  $\mathcal{C}(X_{184})$  :

- $X_{184}$  is the isogonal conjugate of the isotomic conjugate of the circumcenter  $O$  and also the inverse of  $X_{125}$  (center of the Jerabek hyperbola) in the Brocard circle.
- the center of  $\mathcal{C}(X_{184})$  is not mentioned in the current edition of [4]. It is the  $G$ –Ceva conjugate of  $X_{184}$ .
- this conic contains  $X_{112}$  (on the circumcircle),  $X_{248}$ ,  $X_{906}$ ,  $X_{1415}$ ,  $X_{1576}$ ,  $X_{2966}$  (on the Steiner ellipse).
- its axes are parallel to the asymptotes of the rectangular circumhyperbola passing through  $X_{22}$ .
- $\mathcal{C}(X_{184})$  can be construed as the barycentric product of  $O$  and the circumcircle.

### 2.1 Inversible pivotal cubics

A  $p\mathcal{K}$  with pivot  $P$  on the circumcircle, isopivot (or secondary pivot)  $O$ , pole  $\Omega = O \times P$  (barycentric product) on  $\mathcal{C}(X_{184})$  is said to be an invertible pivotal cubic since it is self-inverse in the circumcircle.

The tangents at  $A, B, C, P$  pass through  $O$ . In other words, the circumconic through  $O$  and  $P$  is the polar conic of  $O$  in the cubic.

The singular focus  $F$  of the cubic is the inverse of the antipode of  $cP$  on the nine point circle, where  $cP$  denotes the complement of  $P$ .  $F$  lies on the circumcircle of the tangential triangle i.e. the anticevian triangle of the Lemoine point  $K$ .  $F$  is the Miquel point of the complete quadrilateral formed with the four tangents at  $A, B, C, P$  to the circumcircle.

The cubic contains  $E = \text{gc}P$  (isogonal conjugate of  $cP$ ) and its infinite point is that of the line  $P\text{gc}P$ . In other words, the real asymptote is parallel to  $PE$ .  $E$  also lies on the circumcircle of the tangential triangle.

The inverse  $E'$  of  $E$  lies on the cubic and the  $\Omega$ -isoconjugate  $X$  of  $E'$  is the intersection of the cubic with its real asymptote.  $E'$  lies on the nine point circle and  $X$  is the second intersection of the line  $PE'$  with the circumconic through  $O$  and  $E'$ .

If  $M$  is a point on the cubic, its inverse  $M'$  and its  $\Omega$ -isoconjugate  $M^*$  are on the cubic. The parallel at  $M^*$  to the asymptote meets the line  $PM'$  at  $M''$  on the cubic. This gives as many points as we want on the curve.

### Examples of inversive pivotal cubics

In [2] we find several examples of these cubics namely K112, K113, K114, K336, K436 and also the focal cubics K110, the axial cubics K111.

The figure 1 shows K112 together with the Neuberg cubic and the Lester circle. The two cubics have nine known points in common. K112 contains the inverses of  $X_{13}$ ,  $X_{14}$ ,  $X_{96}$ ,  $X_{265}$  not mentioned in the current edition of [4].

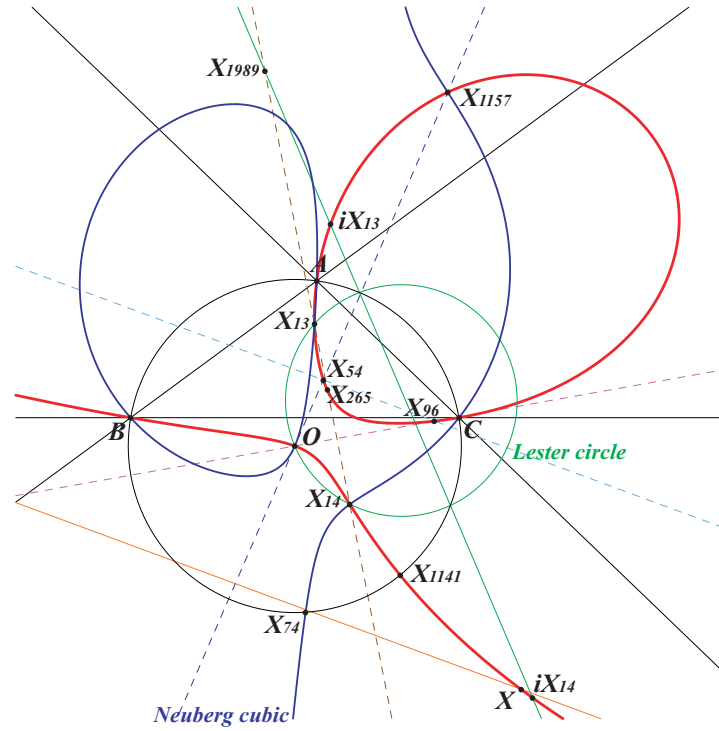


Figure 1: The cubics K112 and K001

## Construction of an inversive pivotal cubic

Let  $M$  be a point on the line  $OP$  and let  $N = M/P$  (Ceva conjugate or cevian quotient). The line  $PN$  meets the circumconic through  $O$  and  $M$  at two isoconjugate points  $U, U^*$  on the cubic. The tangents at these two points meet at  $N^*$  which is the second intersection of the line  $MN$  with the conic above.

Note that the point  $N$  lies on the polar conic of  $P$  which contains  $P$  (with tangent  $OP$ ),  $O/P$  and the vertices of the anticevian triangle of  $P$ . The tangent at  $N$  to this polar conic is the line  $MN$  hence the line  $PN$  is the polar line of  $M$  in this conic.

Remark :  $O$  is the orthocenter of the cevian triangle  $P_aP_bP_c$  of  $P$  and the cubic is also a pivotal cubic with pivot  $O$  (with respect to this triangle) in the isoconjugation with fixed points  $A, B, C, P$ .

## 2.2 Pairs of inverse pivotal cubics

Let us now consider a pole  $\Omega$  on the Brocard axis  $OK$ . The cubic  $\mathcal{K} = p\mathcal{K}(\Omega, P)$  is inverted into another pivotal cubic  $\mathcal{K}' = p\mathcal{K}(\Omega', P')$  if and only if  $P$  is the reflection of  $X_{110}$  (the focus of the Kiepert parabola) in the line  $\mathcal{L}(\text{tg}\Omega)$ , where  $\text{tg}\Omega$  denotes the isotomic conjugate of the isogonal conjugate of  $P$ .

In this case,  $P'$  is the inverse of  $P$  and  $\Omega'$  is the intersection of the Brocard axis with the polar of  $\Omega$  in  $\mathcal{C}(X_{184})$ .

Remarks :

- $\mathcal{L}(\text{tg}\Omega)$  is tangent to the Kiepert parabola  $\mathcal{P}$ . Thus, this is the perpendicular bisector of  $PX_{110}$  and  $P$  lies on the Euler line, the directrix of  $\mathcal{P}$ .
- $\mathcal{L}(\text{tg}\Omega)$  is the  $\Omega$ -isoconjugate of the circumcircle.
- the line  $\Omega\text{tg}\Omega$  is tangent to the inscribed conic with center  $X_{620}$ , the complement of the center  $X_{115}$  of the Kiepert hyperbola. This conic passes through  $X_2, X_{32}, X_{439}, X_{593}, X_{1509}, X_{2482}$ .
- the lines  $\mathcal{L}(\text{tg}\Omega)$  and  $\mathcal{L}(\text{tg}\Omega')$  meet on the tangent at  $X_{110}$  to the circumcircle.

## Points on the cubic $\mathcal{K}$

$\mathcal{K}$  contains the following points :

- $O$ , the pivot  $P$  and  $E = O^*$  ( $\Omega$ –isoconjugate of  $O$ ), these three points on the Euler line. Since  $P$  is the inverse of  $P'$ ,  $\mathcal{K}'$  must contain  $E'$ , the inverse of  $E$  and also the  $\Omega'$ –isoconjugate of  $O$ .
- $P/O$  ( $P$ –Ceva conjugate of  $O$ ) on the Stammler hyperbola.
- the isopivot  $P^*$  on the line  $OP/O$ .  $P^*$  lies on the Jerabek strophoid K039 since its inverse lies on the Jerabek hyperbola (and on  $\mathcal{K}'$ ).
- $S$ , last point on the circumcircle (and on  $\mathcal{K}'$ ).  $S$  is the second intersection of the line  $X_{110} P/O$  with the circumcircle.
- $S^*$  on the line  $PS$  and on  $\mathcal{L}(\text{tg}\Omega)$ .
- $J$ , the third (real) point at infinity, on the line  $P^*S^*$ .
- $J^*$ , on the parallel at  $P$  to  $P^*S^*$ , on the Jerabek hyperbola, on the circumconic through  $P$  and  $S$  (which always contains  $X_{110}$ ).
- $J_6$ , the isogonal conjugate of the inverse of  $P$  or the antipode on the Jerabek hyperbola of the inverse of  $P^*$ . Thus, we know the six common points of  $\mathcal{K}$  and the Jerabek hyperbola.

### Asymptote and singular focus of the cubic $\mathcal{K}$

Let  $P_a, P_b, P_c$  be the vertices of the cevian triangle of  $P$ . Define  $A_2 = AJ^* \cap P_a S^*$  and  $J_a = JA \cap PA_2$ ,  $B_2, C_2, J_b, J_c$  likewise. These nine points lie on  $\mathcal{K}$ . Note that  $A_2$  lies on the circle  $APP^*$ .

The circles  $APJ_a, BPJ_b, CPJ_c$  meet at  $P$  and another point  $X^*$  which is the  $\Omega$ –isoconjugate of the intersection  $X$  of  $\mathcal{K}$  with its real asymptote. This is therefore the parallel at  $X$  to the line  $P^*S^*$ .

$X$  lies on the circle  $A_2B_2C_2$  and its antipode is the singular focus  $F$  of  $\mathcal{K}$ .

### Common points of the inverse cubics $\mathcal{K}$ and $\mathcal{K}'$

These two circular cubics have already seven known common points namely  $A, B, C, O, S$  and the circular points at infinity. They must have two other (not necessarily real) common points  $T, T'$  inverses in the circumcircle hence collinear with  $O$  and lying on the Brocard axis.

The isoconjugation which swaps  $\Omega$  and  $\Omega'$  transforms  $X_6$  into a point  $\omega$ , the pole of the isoconjugation which swaps  $T$  and  $T'$ . Hence, it is sufficient

to intersect the line  $OK$  with its  $\omega$ -isoconjugate to obtain the points  $T$  and  $T'$ .

These two cubics  $\mathcal{K}$  and  $\mathcal{K}'$  generate a pencil of cubics with nine known base points. This pencil contains a third pivotal cubic  $\mathcal{K}''$  which is the inversible  $p\mathcal{K}$  with pivot  $S$  (on the circumcircle) and isopivot  $O$ .

### Examples of inverse pivotal cubics

– the cubics  $K001 = p\mathcal{K}(X_6, X_{30})$  (Neuberg cubic) and its inverse  $K073 = p\mathcal{K}(X_{50}, X_3)$  meet at the isodynamic points  $X_{15}$  and  $X_{16}$ .  $\mathcal{K}''$  is  $K114 = p\mathcal{K}(X_3 \times X_{74}, X_{74})$  containing  $X_3, X_{15}, X_{16}, X_{74}$ . See figure 2.

– the cubics  $K043 = p\mathcal{K}(X_{187}, X_2)$  (Droussent medial cubic) and its inverse  $K108 = p\mathcal{K}(X_{32}, X_{23})$  meet at  $X_6$  and  $X_{187}$ .  $\mathcal{K}''$  is  $p\mathcal{K}(X_3 \times X_{111}, X_{111})$  containing  $X_3, X_6, X_{111}, X_{187}, X_{895}$ .

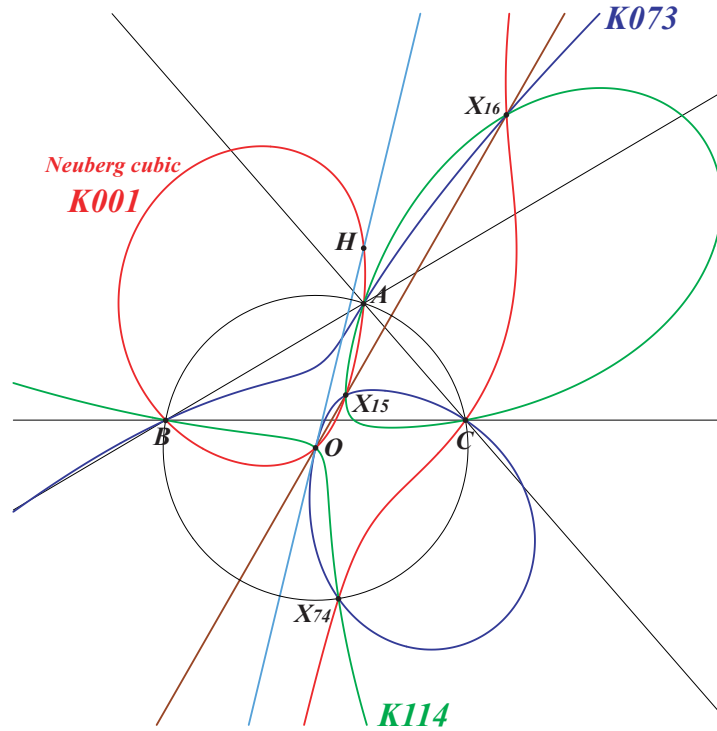


Figure 2: The cubics K001, K073 and K114

### 3 Non-pivotal isocubics

Let  $\mathcal{K} = n\mathcal{K}(\Omega, P, O)$  be a non-pivotal isocubic passing through  $O$  with pole  $\Omega$  and root  $P$  and let  $\mathcal{K}'$  be its inverse in the circumcircle.

**Theorem 2**  *$\mathcal{K}'$  is also a non-pivotal cubic (with pole  $\Omega'$  and root  $P'$ ) if and only if  $\Omega$  lies on the Brocard axis and  $P$  lies on the line  $GK$ .*

#### Points on $\mathcal{K}$

$\mathcal{K}$  contains :

- $X_{110}$ , the focus of the Kiepert parabola  $\mathcal{P}$ ,
- $E_1 = \Omega$ –isoconjugate of  $O$ , on the Euler line,
- $J = \Omega$ –isoconjugate of  $X_{110}$ , the real point at infinity, the infinite point of  $\mathcal{L}(\text{tg}\Omega)$ ,
- $E_2$  on the line  $JX_{110}$  and third point on the Euler line,
- $X = \Omega$ –isoconjugate of  $E_2$ , intersection of the cubic with its real asymptote (which is parallel to  $\mathcal{L}(\text{tg}\Omega)$ ),
- $E_3$  intersection of the line  $OX_{110}$  with the parallel at  $E_1$  to  $\mathcal{L}(\text{tg}\Omega)$ ,
- $E_4 = \Omega$ –isoconjugate of  $E_3$ , intersection of the line  $E_1X_{110}$  with the parallel at  $O$  to  $\mathcal{L}(\text{tg}\Omega)$ ,

#### Properties of $\mathcal{K}$

- The root  $P$  of the cubic is the trilinear pole of the perpendicular bisector of  $E_1, X_{110}$ . This bisector is a tangent to  $\mathcal{P}$  since  $E_1$  is a point on the Euler line.
- $\mathcal{L}(P)$  meets the sidelines of  $ABC$  at three collinear points  $U, V, W$  and the midpoints of  $AU, BV, CW$  lie on a perpendicular to the Euler line.
- The points  $AX_{110} \cap JU$  and  $UX_{110} \cap JA$  lie on  $\mathcal{K}$ , four other points similarly.
- The tangents to  $\mathcal{K}$  at  $O, X_{110}, J$  (the asymptote),  $E_1$  concur at  $X$ . In other words, the polar conic of  $X$  is the conic passing through  $X, O, X_{110}, J, E_1$ .
- The polar conic of the infinite point  $J$  is a rectangular hyperbola which contains the midpoint of any two points on a parallel to the asymptote e.g.  $O - E_4, X_{110} - E_2, E_1 - E_3$ , etc.
- The points  $X, O, X_{110}, E_1$  are concyclic on a circle with center the intersection of the perpendicular bisector of  $O, X_{110}$  and  $\mathcal{L}(P)$ . The singular focus  $F$  of  $\mathcal{K}$  is the antipode of  $X$  on this circle.  $F$  can also be construed

as  $\text{igsi}E_1$  where  $i$ ,  $g$ ,  $s$  are the inversion in the circumcircle, the isogonal conjugation, the reflection in the nine point center  $X_5$  respectively.

This give the following

**Theorem 3**  $\mathcal{K}$  is a pivotal isogonal circular cubic with pivot  $J$  (the infinite point of  $\mathcal{L}(\text{tg}\Omega)$  with respect to the triangle  $OX_{110}E_1$ ).

It immediately follows that  $\mathcal{K}$  contains the in/excenters of this triangle.

The figure 3 shows  $K022 = n\mathcal{K}(X_3, X_{524}, X_3)$  which is an example of such cubic. Here,  $E_1 = G$  hence K022 is an isogonal circular pivotal cubic with pivot  $X_{525}$  with respect to the triangle  $GOX_{110}$ . The inverse of K022 is  $n\mathcal{K}(X_3 \times X_{23}, X_6, X_3)$ .

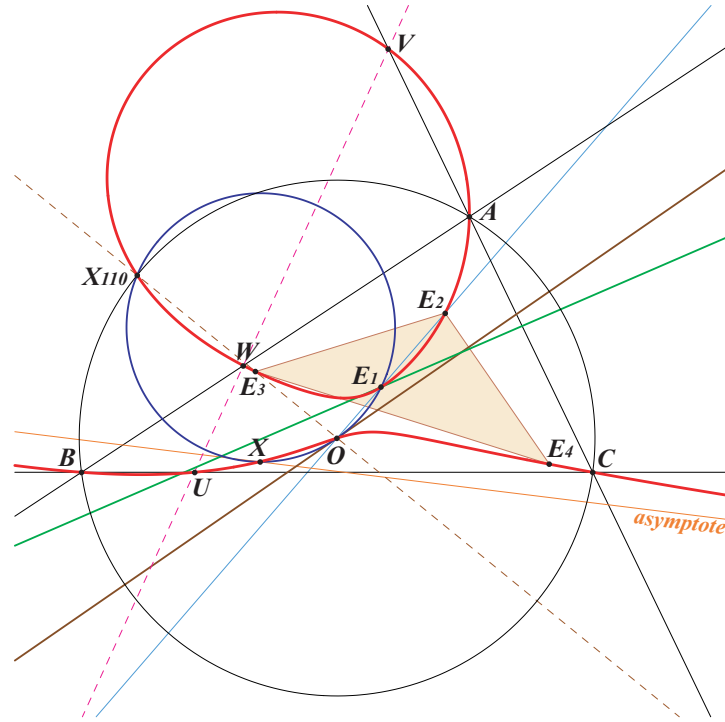


Figure 3: The cubic K022

### The cubics $\mathcal{K}$ and $\mathcal{K}'$

The properties seen above ensure a simultaneous construction of the elements of these two inverse cubics.



- Let  $T$  be a point on the tangent at  $X_{110}$  to the circumcircle such that  $T$  lies outside the parabola  $\mathcal{P}$ .
- Let  $\mathcal{T}$  and  $\mathcal{T}'$  be the two tangents drawn through  $T$  to  $\mathcal{P}$ .
- The roots  $P$  and  $P'$  of  $\mathcal{K}$  and  $\mathcal{K}'$  are the trilinear poles of these tangents, these points lying on the line  $GK$ .
- The reflections  $E_1$  and  $E'_1$  of  $X_{110}$  in  $\mathcal{T}$  and  $\mathcal{T}'$  are two points on the Euler line (and on the cubics) which are the isoconjugates of  $O$  in the isoconjugations with poles  $\Omega$  and  $\Omega'$  respectively.
- Hence  $\Omega$  and  $\Omega'$  are the barycentric products  $O \times E_1$  and  $O \times E'_1$  respectively.

Remarks :

- If  $T$  is an intersection of the tangent at  $X_{110}$  to the circumcircle with  $\mathcal{P}$ , one can draw only one (double) tangent to  $\mathcal{P}$  hence  $\mathcal{K}$  and  $\mathcal{K}'$  would coincide but, in this case,  $E_1$  would be one of the common points of the Euler line and the circumcircle. This gives seven points of the cubic on the circumcircle and the cubic must decompose.

It follows that there is no inversive non degenerate  $n\mathcal{K}$  contrary to what we had for the  $p\mathcal{K}$  cubics.

- $\mathcal{K}$  and  $\mathcal{K}'$  are  $n\mathcal{K}_0$  cubics (without term in  $xyz$ ) if and only if the two tangents  $\mathcal{T}$  and  $\mathcal{T}'$  are those passing through  $X_{1576} = \text{gt}X_{110}$  but, since this point is not always outside the parabola  $\mathcal{P}$ , there are not always such cubics. This is the case only when the triangle is obtuse angle.

## References

- [1] J.P. Ehrmann and B. Gibert, *Special Isocubics in the Triangle Plane*, available at <http://perso.orange.fr/bernard.gibert/>
- [2] B. Gibert, *Cubics in the Triangle Plane*, available at <http://perso.orange.fr/bernard.gibert/>
- [3] C. Kimberling, Triangle Centers and Central Triangles, *Congressus Numerantium*, 129 (1998) 1–295.
- [4] C. Kimberling, *Encyclopedia of Triangle Centers*, <http://www2.evansville.edu/ck6/encyclopedia>.