

On a Class of Focal Cubics

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1 Introduction

In the plane of the usual reference triangle ABC , we consider two fixed points $P = p : q : r$ and $U = u : v : w$ (barycentric coordinates) not lying on the sidelines of ABC . If M is a variable point, we reflect the line AM in a bisector of $\angle AP, AU$ to obtain the line L_A and we define L_B, L_C similarly.

We shall consider that the union of the bisectors of $\angle AP, AU$ is a degenerate rectangular hyperbola $\mathcal{H}_A, \mathcal{H}_B$ and $\mathcal{H}_C$ being defined likewise.

We wish to find the locus of M such that the three lines L_A, L_B, L_C are concurrent (say at M').

One obvious and trivial situation is obtained when the points P and U are isogonal conjugates. Indeed, the rectangular hyperbola $\mathcal{H}_A$ is the union of the bisectors at A in the triangle ABC and thus, the three hyperbolas belong to a same pencil of rectangular hyperbolas passing through the in/excenters of ABC . In this case, the lines L_A, L_B, L_C are concurrent for any point M and M' is the isogonal conjugate of M . M' is the pole of M in the pencil i.e. the polar lines of M in all the rectangular hyperbolas of the pencil concur at M' .

This special situation will be excluded in the sequel i.e. we will suppose that P and U are not isogonal conjugates.

At last, note that P and U are not necessarily distinct. In this case, L_A is simply the reflection of the line AM in the line AP . This will give a special situation we shall study below.

We shall use the notations :

- M^* or $\mathbf{g}M$ is the isogonal conjugate of M (with respect to ABC),
- $\mathbf{i}M$ is the inverse of M in the circumcircle of ABC ,
- $\mathbf{gig}M$ is the antigonal of M .

2 The focal cubic $\mathcal{C}(P, U)$

2.1 First definition and consequences

Theorem 1. *The lines L_A, L_B, L_C are concurrent (at M') if and only if M lies on a focal cubic $\mathcal{C}(P, U)$. In this case, M' also lies on $\mathcal{C}(P, U)$ and is called the conjugate of M on $\mathcal{C}(P, U)$.*

Proof. The polar of M in $\mathcal{H}_A$ is the line L_A hence L_A, L_B, L_C are concurrent if and only if M lies on the jacobian of the net of rectangular hyperbolas generated by $\mathcal{H}_A, \mathcal{H}_B$ and $\mathcal{H}_C$. This jacobian is a focal cubic since each conic of the net is a rectangular hyperbola. $\square$

Immediate consequences

- $\mathcal{C}(P, U) = \mathcal{C}(U, P)$.
- $\mathcal{C}(P, U)$ passes through A, B, C hence it is a circumcubic. Indeed, if $M = A$, the line L_A is not defined thus L_B and L_C are concurrent.

Equivalently, the jacobian of a net of conics must contain the centers of all the degenerate conics of the net. This is the case of $\mathcal{H}_A, \mathcal{H}_B$ and $\mathcal{H}_C$.

- $\mathcal{C}(P, U)$ contains P and U and these points are conjugate on the cubic.
- If P and U are distinct points on the line at infinity $\mathcal{L}^\infty$, $\mathcal{C}(P, U)$ must decompose since it has at least four points on $\mathcal{L}^\infty$ namely P, U and the circular points. It becomes the union of $\mathcal{L}^\infty$ and a rectangular circum-hyperbola.

If P and U are distinct points on the circumcircle $(\mathcal{O})$ of ABC , $\mathcal{C}(P, U)$ must also decompose since it has at least seven points on $(\mathcal{O})$ namely A, B, C, P, U and the circular points. It becomes the union of $(\mathcal{O})$ and the perpendicular bisector of PU .

If P lies on $\mathcal{L}^\infty$ and U lies on $(\mathcal{O})$ then $\mathcal{C}(P, U)$ is the union of $\mathcal{L}^\infty$ and $(\mathcal{O})$.

These three special cases are excluded in the sequel.

- M lies on $\mathcal{C}(P, U)$ if and only if U lies on $\mathcal{C}(P^*, M)$.
- The isogonal transform (with respect to ABC) of $\mathcal{C}(P, U)$ is $\mathcal{C}(P, U)^* = \mathcal{C}(P^*, U^*)$.
- When $U = \mathbf{i}P$, the bisectors $\mathcal{H}_A, \mathcal{H}_B$ and $\mathcal{H}_C$ have two common points on the circumcircle and on the line PU . It follows that $\mathcal{C}(P, U)$ has at least four points on the line PU (which contains X_3) and must decompose into this line and the circumcircle.

For example, $\mathcal{C}(X_4, X_{186})$ is the union of the circumcircle and the Euler line.

- Consequently, by isogonal conjugation, when $U = \mathbf{gig}P$, $\mathcal{C}(P, U)$ must also decompose into the line at infinity and the rectangular circum-hyperbola passing through P .

For example, $\mathcal{C}(X_6, X_{67})$ is the union of the line at infinity and the Jerabek hyperbola.

Points on the focal cubic $\mathcal{C}(P, U)$

- A', B', C' conjugates of A, B, C .

The line L_A is not defined. A' is the intersection of the reflections of AB in the bisectors at B and AC in the bisectors at C .

- A_1, B_1, C_1 on the sidelines of ABC and A'_1, B'_1, C'_1 their conjugates on the cubic. The reflections of BC in the bisectors at B and C are the lines BC' and CB' . They meet at A'_1 on the cubic and the conjugate A_1 of A'_1 is the intersection of BC and $B'C'$.

Hence we have now the following collinearities on $\mathcal{C}(P, U)$:

$$A', B, C'_1; \quad A'CB'_1; \quad A'B'C_1; \quad A'C'B_1,$$

and the others similarly.

- We can start this again and again with $A_2 = B_1C_1 \cap B'_1C'_1$ and its conjugate $A'_2 = B_1C'_1 \cap B'_1C_1$, and, each time, we have six more points on the cubic.
- P_A, P_B, P_C on the cevians of P and U_A, U_B, U_C on the cevians of U .
 $P_A = AP \cap A'U$ and its conjugate $U_A = AU \cap A'P$ obviously lie on $\mathcal{C}(P, U)$.
- Q on the line PU and its conjugate Q' .

It is known that the tangents at any two conjugate points on the cubic meet at another point on the cubic. In particular, the tangents at P, U meet at Q' which is the fourth common point of the conics $PUABC'_1$ and $PUACB'_1$. Q is the intersection of the line PU and the reflection of AQ' in a bisector of $\angle AP, AU$.

2.2 Second definition and consequences

Lemma 1. *For any point M on $\mathcal{C}(P, U)$, its conjugate M' is in fact the isogonal conjugate of M in the triangle BCA'_1 .*

Proof. We know that the line BC' is the reflection of BC in the bisectors of $\angle(BP, BU)$. But BC' contains A'_1 thus these bisectors are those at B in the triangle BCA'_1 . Similarly, the bisectors of $\angle(CP, CU)$ are those at C in the triangle BCA'_1 . This completes the proof. $\square$

Naturally, M' is also the isogonal conjugate of M in the triangles CAB'_1 and ABC'_1 .

Theorem 2. *$\mathcal{C}(P, U)$ is an isogonal non-pivotal cubic (a $n\mathcal{K}$) in the triangles BCA'_1, CAB'_1 and ABC'_1 .*

Proof. Following the lemma above, $\mathcal{C}(P, U)$ is an isogonal circumcubic in BCA'_1 but since it meets the sidelines of BCA'_1 at three collinear points B', C', A_1 , it must be a $n\mathcal{K}$. $\square$

This gives several corollaries which are as many other characterizations of $\mathcal{C}(P, U)$.

Corollary 1. *If R, S are any two (non conjugate) points on $\mathcal{C}(P, U)$ and if R', S' are their conjugates, $\mathcal{C}(P, U)$ is the locus of point M such that the angles of lines MR, MS and MR', MS' are equal (mod. π).*

For example, for any point M on $\mathcal{C}(P, U)$, the segments AU and $A'P$ (or BC and $B'C'$) are seen from M under equal line angles.

It follows that $\mathcal{C}(P, U)$ also contains $T = RS \cap R'S'$, its conjugate $T' = R'S \cap RS'$. The third point T'' on TT' is the orthogonal projection of $RR' \cap SS'$ on TT' . The third points R'' on RR' and S'' on SS' are defined similarly.

Corollary 2. *$\mathcal{C}(P, U)$ is the locus of foci on conics inscribed in the quadrilateral formed by the lines $B'C'A_1, BCA_1, BC'A'_1$ and $B'CA'_1$.*

Corollary 3. *Two conjugate points on $\mathcal{C}(P, U)$ have their midpoint on a fixed line which is called the axis or orthic line of the focal cubic.*

Remark : this axis is the Newton line of the quadrilateral above and also the locus of points whose polar conic is a rectanguler hyperbola and the Laplacian of the cubic.

For example, the midpoints of $PU, AA', BB', CC', A_1A'_1$, etc, P_AU_A , etc, are collinear.

2.3 Infinite point and singular focus of $\mathcal{C}(P, U)$

Theorem 3. *The singular focus F of $\mathcal{C}(P, U)$ is the intersection of the circles BCA'_1 , CAB'_1 , ABC'_1 . The (real) infinite point J is its conjugate and thus, the infinite point of the axis of the cubic.*

Proof. $\mathcal{C}(P, U)$ being an isogonal cubic with respect to BCA'_1 , its singular focus must lie on the circumcircle of BCA'_1 . For the same reasons, it must lie on the two other circles. $\square$

Remark : it follows that there is only one point of the plane such that the lines L_A , L_B , L_C are parallel and this point is F .

More generally, following the notations of Corollary 1, F is the Miquel point of the quadrilateral $RSR'S'$ hence its lies on the circles TRS' , $TR'S$, $T'RS$, $T'R'S'$ and also $R''S''T''$.

Theorem 4. *The singular foci of $\mathcal{C}(P, U)$ and $\mathcal{C}(P, U)^*$ are inverse in the circumcircle of ABC .*

Proof. It is sufficient to observe that the singular focus F' of $\mathcal{C}(P, U)^*$ must lie on the circle $BCgA'$ which is the inverse of the circle BCA'_1 passing through F . See §2.6 below. $\square$

2.4 Axis and asymptote of $\mathcal{C}(P, U)$

According to known properties of focal cubics, the real asymptote of $\mathcal{C}(P, U)$ is the image of its axis under the homothety with center F and ratio 2. Indeed, we have said that the midpoint of the two conjugates F and J must lie on the axis.

Following the same idea, $\mathcal{C}(P, U)$ meets its real asymptote at X whose conjugate X' is the third intersection of the parallel at F to the asymptote with the cubic.

Hence, the ‘conjugate transform’ of this latter line is a hyperbola passing through B , C , A'_1 , J , F and naturally X . This gives an easy (ruler only) construction of X :

BC meets FJ at P_1 , CA'_1 meets the asymptote at P_2 , the line P_1P_2 meets FA'_1 at P_3 and then X is the intersection of BP_3 with the asymptote.

Remark 1 : the line FX is the tangent at F to the cubic.

Remark 2 : the polar conic γ_F of F is the circle centered on the perpendicular at F to FX and on the perpendicular at X' to FX' . This circle passes through F and X' . γ_F meets the axis at two (real or not, distinct or not) points E_1 , E_2 which lie on the cubic. The tangents at these points pass through F . More precisely,

- if E_1 , E_2 are real and distinct, $\mathcal{C}(P, U)$ has one circuit,
- if E_1 , E_2 are imaginary, $\mathcal{C}(P, U)$ has two circuits,
- if $E_1 = E_2$, $\mathcal{C}(P, U)$ is a nodal (unicursal) cubic therefore a strophoid.

2.5 Construction of $\mathcal{C}(P, U)$

The knowledge of F , X , X' gives an easy construction of $\mathcal{C}(P, U)$.

Draw a circle passing through F and X' with center ω . The perpendicular at X to the line $F\omega$ meets the circle at two points M , M_1 on $\mathcal{C}(P, U)$.

Hence, the third point M_1 of any line XM on the cubic is the reflection of M in the perpendicular at F to the line MX . The reflection of the parallel at M to the asymptote in the axis meets FM at N and $X'M_1$ at M' , the conjugate of M .

2.6 Concyclicities on $\mathcal{C}(P, U)$

2.6.1 Circles passing through F

Theorem 5. *Any circle passing through F and a point M on $\mathcal{C}(P, U)$ meets the cubic again at two points collinear with M' , the conjugate of M .*

Proof. M' is the (Sylvester) coresidual of F , M and the circular points at infinity. $\square$

Conversely and obviously, if a line meets $\mathcal{C}(P, U)$ at three points M, N, Q then the circle FMN must contain Q' , the conjugate of Q .

This shows that F lies on many other circles defined from the points we have met before.

- A' lies on the lines $BC'_1, CB'_1, B'C_1, C'B_1$ hence F lies on the circles ABC'_1, ACB'_1 (both already mentioned), $AB'C_1, AC'B_1$.
- A_1 lies on $B'C'$ thus F lies on the circle $B'C'A'_1$.
- A lies on the lines BC_1, CB_1, PP_A, UU_A hence F lies on the circles $A'BC_1, A'B_1C$ (both already mentioned), $A'PP_A, A'UU_A$.

Corollary 4. *Any circle passing through F meets $\mathcal{C}(P, U)$ again at three points such that $\mathcal{C}(P, U)$ is an isogonal $n\mathcal{K}$ with respect to the triangle formed by these three points.*

2.6.2 Circles with center F

Any circle with center F must meet the cubic at two points on a line passing through X . Conversely, if two points of the cubic are equidistant of F , the line passing through them must contain X . All this derives of the construction given above.

2.6.3 Circumcircles of triangles inscribed in $\mathcal{C}(P, U)$

Theorem 6. *If a circle meets $\mathcal{C}(P, U)$ at three known points M_1, M_2, M_3 with conjugates M'_1, M'_2, M'_3 then it meets the cubic at a fourth point which must also lie on the circles $M_1M'_2M'_3, M'_1M_2M'_3, M'_1M'_2M_3$.*

Proof. Any circle passing through M_2, M_3 meets the cubic again at two points collinear with a point Q_1 which only depends of M_2, M_3 . If the circle degenerates into the union of the line M_2M_3 and the line at infinity, the two points are J and $N_1 = M_2M_3 \cap M'_2M'_3$. If the circle is FM_2M_3 , the two points are F and $N'_1 = M_2M'_3 \cap M'_2M_3$, the conjugate of N_1 . Thus, Q_1 is the intersection of the line FN'_1 and the parallel at N_1 to the asymptote.

A similar proof shows that any circle passing through M'_2, M'_3 meets the cubic again at two points which are also collinear with the same point Q_1 . Hence the circles $M_1M_2M_3$ and $M_1M'_2M'_3$ have in common M_1 and another point on the cubic and on the line M_1Q_1 . $\square$

Corollary 5. *If we know a circle passing through three points on the cubic, we can obtain three other circles by conjugating two out of the three points and then these four circles pass through a same point on the cubic.*

2.6.4 More points on $\mathcal{C}(P, U)$

As an application of the previous corollary, we obtain pairs of conjugate points on $\mathcal{C}(P, U)$ which are obtained from the already known points and which all lie on several circles.

The points S_i have their origin in a triangle $A_i B_i C_i$ and the circles are obtained by the conjugation of two points (we assume that $ABC = A_0 B_0 C_0$). Recall that A_{i+1} is the third point of the cubic on the line $B_i C_i$ and also on the line $B'_i C'_i$.

The points S'_i are the conjugates of the points S_i . A'_{i+1} is the third point of the cubic on the lines $B'_i C_i$ and $B_i C'_i$.

There are always four circles passing through a point S_i or S'_i .

The points $S_{i,j}$ have their origin in a triangle $A_i B_j C_j$. There are twelve circles passing through each of these points. The conjugate $S'_{i,j}$ of $S_{i,j}$ is a point of same nature with the triangle $A'_i B'_j C'_j$.

At last, there are no circles with the “triangles” $A_{i+1} B_i C_i$, $A'_{i+1} B'_i C'_i$, $A_{i+1} B_i C'_i$ since the points are collinear.

Table 1: Concyclicities on $\mathcal{C}(P, U)$.

this point	lies on the circles
S_0	$ABC, AB'C', A'BC', A'B'C$
S'_0	$A'B'C', A'BC, AB'C, AB'C$
S_1	$A_1 B_1 C_1, A_1 B'_1 C'_1, A'_1 B_1 C'_1, A'_1 B'_1 C_1$
S'_1	$A'_1 B'_1 C'_1, A'_1 B_1 C_1, A_1 B'_1 C_1, A_1 B_1 C'_1$
$S_{0,1}$	$AB_1 C_1, AB'_1 C'_1, A_1 B' C'_1$ and nine similar circles
$S'_{0,1}$	$A' B'_1 C'_1, A' B_1 C_1, AB_1 C'_1$ and nine similar circles

Remark : this shows that we are able to find infinitely many points on the cubic by collinearities with the points A_i, B_i, C_i using the ruler only and by concyclicities with the points S_i or $S_{i,j}$ using the ruler and the compass.

3 Remarkable examples of $\mathcal{C}(P, U)$

The tables in the annexe at the end give a selection of these focal cubics with a possible reference in [4]. When they are distinct, P and U are chosen with the lowest possible position in [7]. Indeed, we must not forget that if P' and U' are two conjugate points on $\mathcal{C}(P, U)$ then $\mathcal{C}(P', U') = \mathcal{C}(P, U)$. For example $K300 = \mathcal{C}(X_4, X_6) = \mathcal{C}(X_{23}, X_{265}) = \mathcal{C}(X_{511}, X_{671})$, etc.

3.1 The strophoids $\mathcal{C}(P, P)$, see table in §6.1

$\mathcal{C}(P, P)$ is always a strophoid with node P and these strophoids form the class CL038 in [4] where detailed explanations and properties are given.

The converse is not true : one can find distinct points P, U such that $\mathcal{C}(P, U)$ is a strophoid. For example, $K039 = \mathcal{C}(X_3, X_3)$ (Jerabek strophoid) is also $\mathcal{C}(X_{74}, X_{186})$, $\mathcal{C}(X_{187}, X_{316}^*)$, etc.

When P lies on the circumcircle or on the line at infinity, the strophoid decomposes as seen in §2.1.

Remarks :

- The strophoid $\mathcal{C}(P, P)$ contains the circumcenter X_3 when P lies on the inversive circular quartic Q023 in [4] or on the circumcircle in which case it decomposes.
- The strophoid $\mathcal{C}(P, P)$ contains the orthocenter X_4 when P lies on the antigonal circular quintic Q038 in [4] or on the line at infinity in which case it decomposes. Q038 is the inverse of Q023 in the circumcircle of ABC .

3.2 The focals $\mathcal{C}(X_1, U)$, see table in §6.2

These cubics form a family which is invariant under isogonal conjugation. Indeed, the isogonal transform of $\mathcal{C}(X_1, U)$ is $\mathcal{C}(X_1, U^*)$ (assuming that U is not X_1 or an excenter).

These pairs of cubics are denoted by the symbols of the form (x) and (x)* in the table.

The frequent apparition of the points X_{36} and X_{80} is an application of one of the immediate consequences in §2.1.

- $\mathcal{C}(X_1, U)$ contains X_{36} if and only if U lies on $\mathcal{C}(X_1, X_{80})$ which is the union of the circumcircle and the line X_1X_3 . This line also contains X_{35} , X_{40} , X_{55} , X_{484} , X_{517} , etc.

- $\mathcal{C}(X_1, U)$ contains X_{80} if and only if U lies on $\mathcal{C}(X_1, X_{36})$ which is the union of the line at infinity and the Feuerbach hyperbola. This contains X_4 , X_{79} , X_{84} and many other points.

- Note that $\mathcal{C}(X_1, X_{104})$ and $\mathcal{C}(X_1, X_{517})$ both contain X_{36} and X_{80} since X_{104} lies on the circumcircle and on the Feuerbach hyperbola and X_{517} is the infinite point of the line X_1X_3 .

$\mathcal{C}(X_1, X_{104}) = \mathcal{C}(X_3, X_{80})$ is the inverse of $\mathcal{C}(X_1, X_3)$ and $\mathcal{C}(X_1, X_{517}) = \mathcal{C}(X_4, X_{36})$ is the isogonal transform of $\mathcal{C}(X_1, X_{104})$.

3.3 The focals $\mathcal{C}(X_2, U)$ and $\mathcal{C}(X_6, U^*)$, see table in §6.3

We suppose U distinct of the Lemoine point X_6 . The isogonal transform of $\mathcal{C}(X_2, U)$ is $\mathcal{C}(X_6, U^*)$.

Here again, the frequent occurrences of X_{67} come from the fact that $\mathcal{C}(X_6, X_{67})$ is the union of the line at infinity and the Jerabek hyperbola. In other words, each $\mathcal{C}(X_2, U)$ with U on one of these two latter curves must contain X_{67} .

Similarly, any $\mathcal{C}(X_2, U)$ with U on the Brocard axis or on the circumcircle must contain X_{187} .

3.4 The focals $\mathcal{C}(X_3, U)$ and $\mathcal{C}(X_4, U)$, see table in §6.4

These are probably the most interesting cubics of this type. Indeed, $\mathcal{C}(X_3, U)$ always contains the circumcenter X_3 hence its inversive image in the circumcircle is another focal cubic passing through X_3 .

We suppose U distinct of the orthocenter X_4 . The singular focus of $\mathcal{C}(X_3, U)$ is F , the inverse of the antigonal of U i.e. $F = \mathbf{igig}U$. Apart X_3 and F , $\mathcal{C}(X_3, U)$ also contains U and $\mathbf{ig}U$.

When U lies at infinity, $\mathcal{C}(X_3, U)$ is the union of the circumcircle and a line passing through X_3 .

When U does not lie on the circumcircle, the inverse of $\mathcal{C}(X_3, U)$ is a cubic of the same type namely $\mathcal{C}(X_3, V)$ where V is the antigonal of U . In such case, V is not the orthocenter X_4 . Note that $\mathcal{C}(X_3, V)$ contains U^* and its focus is $\mathbf{i}U$.

The isogonal transform of $\mathcal{C}(X_3, U)$ is $\mathcal{C}(X_4, U^*)$. We recall that P lies on $\mathcal{C}(X_3, U)$ if and only if U lies on $\mathcal{C}(X_4, P)$.

For example, the inverse of $\mathcal{C}(X_3, X_5)$ is $\mathcal{C}(X_3, X_{1263})$, its isogonal transform is $\mathcal{C}(X_4, X_{54})$ and the isogonal transform of $\mathcal{C}(X_3, X_{1263})$ is $\mathcal{C}(X_4, X_{1157})$.

When U lies on the circumcircle, the antipodal of U is always X_4 and the focus of $\mathcal{C}(X_3, U)$ is X_{186} , the inverse of X_4 . The inverse of $\mathcal{C}(X_3, U)$ is of different type since V as above would be X_4 which cannot be acceptable.

A direct computation shows that the inverse of $\mathcal{C}(X_3, U)$ (with U on the circumcircle) is an isogonal $n\mathcal{K}$ whose root lies on the trilinear polar of X_{264} , the isotomic conjugate of X_3 . This cubic is a focal cubic with focus U . It is the locus of foci of inconics centered on a line passing through X_5 , the nine point center.

For example, with $U = X_{98}$ (Tarry point), we obtain the cubic $\mathcal{C}(X_3, X_{98})$ whose inverse is K433 = $n\mathcal{K}(X_6, X_{850}, X_3)$. The line passing through X_5 is parallel to the Brocard axis.

4 Central $\mathcal{C}(P, U)$

In the tables, we meet several central cubics such as K065 = $\mathcal{C}(X_2, X_{524})$. In this section, we want to characterize all the cubics $\mathcal{C}(P, U)$ having a center of symmetry which must be the singular focus F . The conjugate of F being the infinite point J of the cubic, we have $\mathcal{C}(P, U) = \mathcal{C}(F, J)$.

We will suppose that F does not lie at infinity or on the circumcircle and is distinct of X_3 . Indeed, all these cases give decomposed cubics.

Let then $F = p : q : r$ be the singular focus of the sought cubic. An easy calculation gives the following theorem :

Theorem 7. *For any point F not lying at infinity or on the circumcircle and distinct of X_3 , the cubic $\mathcal{C}(F, J)$, where J is the infinite point of the line passing through F and $\text{cg}F$ ¹, is a central cubic with center F .*

This central cubic also contains $Q = \text{ig}F$, its conjugate Q' and their reflections S, S' in F . We have $S = FQ \cap JQ'$ and $S' = FQ' \cap JQ$ which are also conjugate points on the cubic hence $\mathcal{C}(F, J) = \mathcal{C}(Q, Q') = \mathcal{C}(S, S')$.

Conversely, if the infinite point J is given, we find that the focus F must lie on a circular circum-cubic $\mathcal{K}_{\mathcal{J}}$ which contains X_3 , the vertices of the medial triangle, J and J^* (on the circumcircle). The singular focus is the midpoint of X_3J^* . The last point E on the nine point circle is the complement of the cevian product (cevapoint) of X_3 and J . This point E is the coresidual of A, B, C, X_3 .

This shows that all these cubics $\mathcal{K}_{\mathcal{J}}$ are in a same pencil of cubics and their singular foci lie on the circle with center X_3 , radius half that of the circumcircle of ABC .

The real asymptote of $\mathcal{K}_{\mathcal{J}}$ is the Simson line of the antipode of J^* on the circumcircle. Thus, when J traverses the line at infinity, its envelope is the Steiner deltoid $\mathcal{H}_3$. The sixth point of $\mathcal{K}_{\mathcal{J}}$ on the circum-conic through J and J^* is the point X , intersection of the cubic with its real asymptote. $\mathcal{K}_{\mathcal{J}}$ also contains $X_3J \cap EJ^*$.

For example, when $J = X_{524}$ (infinite point of the line X_2X_6), $\mathcal{K}_{\mathcal{J}}$ is the Droussent medial cubic K043 which is the only $p\mathcal{K}$ of the pencil. K043 contains $X_2, X_6, X_{67}, X_{187}, X_{468}, X_{1560}, X_{2482}$ (and X_3, X_{111}, X_{524}) giving as many central cubics. Among them, the most interesting are K065 = $\mathcal{C}(X_2, X_{524})$, K435 = K063* = $\mathcal{C}(X_{187}, X_{524}) =$

¹ $\text{cg}F$ is the complement of the isogonal conjugate of F .

$\mathcal{C}(X_2, X_{843})$ and $\mathcal{C}(X_6, X_{524})$ already mentioned in the tables given in the annexe at the end.

Other central cubics : $\mathcal{C}(X_1, X_{519})$, $\mathcal{C}(X_4, X_{30})$, $\mathcal{C}(X_4, X_{1157})$, $\mathcal{C}(X_9, X_{527})$, $\mathcal{C}(X_{10}, X_{758})$, $\mathcal{C}(X_{11}, X_{521})$.

5 Additional remarks on the net of rectangular hyperbolas

Recall that the degenerate rectangular hyperbola $\mathcal{H}_A$ is the union of the bisectors of $\angle AP, AU$, $\mathcal{H}_B$ and $\mathcal{H}_C$ being defined likewise. We suppose P and U distinct.

These hyperbolas generate the net $\mathcal{H}$ and each conic of $\mathcal{H}$ is a rectangular hyperbola. In particular, $\mathcal{H}$ contains the degenerate hyperbola which is the union of $\mathcal{L}^\infty$ and the axis of $\mathcal{C}(P, U)$.

In general, the net $\mathcal{H}$ does not contain a circum-hyperbola in ABC .

A straightforward computation shows that this happens when U lies on a circum-cubic $\mathcal{K}(P)$ passing through P^* (isogonal conjugate of P in ABC) and $P^\#$ (isogonal conjugate of P in the orthic triangle of ABC).

Furthermore, $\mathcal{K}(P)$ is a pivotal cubic (a $p\mathcal{K}$) if and only if P lies on the Neuberg cubic. The most interesting example is $K434 = \mathcal{K}(X_1) = p\mathcal{K}(X_6, X_{1770})$ passing through $X_1, X_{35}, X_{46}, X_{79}, X_{90}, X_{191}, X_{267}, X_{1717}, X_{1770}, X_{1780}$.

For example, with $P = X_1$ and $U = X_{35}$, the net $\mathcal{H}$ contains the Feuerbach hyperbola. With $U = X_{79}$ or X_{191} , it contains the Kiepert hyperbola.

6 Annexe : tables of cubics

Explanation of the notes mentioned in the tables :

- (1) Other definitions are given below.
- (2) The axis of the cubic is the Euler line. More generally, the axis of any cubic $\mathcal{C}(X_4, U)$ with U on the Euler line is the Euler line.
- (3) The axis of the cubic is the Brocard line.
- (4) This is a Droz-Farny cubic of the class CL039 in [4].
- (5) This is a central cubic.

6.1 Strophoids

Table 2: A selection of remarkable strophoids $\mathcal{C}(P, P)$

P	centers on $\mathcal{C}(P, U)$	cubic	notes
X_2	X_2, X_{468}, X_{671}		
X_3	$X_3, X_{74}, X_{186}, X_{187}, X_{1157}$	K039	
X_4	$X_4, X_{30}, X_{265}, X_{316}, X_{671}, X_{1263}, X_{1300}$	K025	(2)(4)
X_5	X_4, X_5, X_{252}		
X_6	X_6, X_{187}, X_{895}		
X_{13}	X_{13}, X_{14}, X_{530}	K061a	
X_{14}	X_{13}, X_{14}, X_{531}	K061b	
X_{36}	$X_3, X_{36}, X_{186}, X_{859}, X_{953}$	K274	(1)
X_{54}	X_3, X_{54}, X_{143}		
X_{80}	$X_4, X_{80}, X_{150}, X_{265}, X_{952}$	K275	(1)

Notes : (1) K274 and K275 are also $\mathcal{C}(X_3, X_{953})$ and $\mathcal{C}(X_4, X_{952})$ respectively. The inverse of K274 is K165.

6.2 $\mathcal{C}(X_1, U)$

Table 3: A selection of remarkable focals $\mathcal{C}(X_1, U)$

U	centers on $\mathcal{C}(X_1, U)$	cubic	note
X_3	$X_1, X_3, X_{36}, X_{104}$		(a)
X_4	$X_1, X_4, X_{80}, X_{355}, X_{517}$		(a)*
X_7	$X_1, X_7, X_{80}, X_{518}, X_{1002}, X_{1155}$		(b)
X_{35}	$X_1, X_{21}, X_{35}, X_{36}, X_{60}, X_{501}, X_{759}, X_{3065}$		(c)
X_{40}	$X_1, X_{36}, X_{40}, X_{90}, X_{102}, X_{1785}$		(d)
X_{55}	$X_1, X_{36}, X_{55}, X_{105}, X_{1001}, X_{1156}$		(b)*
X_{79}	$X_1, X_{12}, X_{65}, X_{79}, X_{80}, X_{484}, X_{502}, X_{758}, X_{2475}$		(c)*
X_{84}	$X_1, X_{46}, X_{80}, X_{84}, X_{515}, X_{1158}, X_{1795}, X_{2077}$		(d)*
X_{104}	$X_1, X_3, X_{36}, X_{80}, X_{104}, X_{952}$		(e)
X_{484}	$X_1, X_{36}, X_{79}, X_{267}, X_{484}, X_{501}, X_{3065}$		(f)
X_{515}	$X_1, X_{80}, X_{84}, X_{515}, X_{1785}, X_{2716}$		
X_{516}	$X_1, X_{80}, X_{516}, X_{1323}, X_{2448}, X_{2449}, X_{2717}, X_{3062}$		
X_{517}	$X_1, X_4, X_{36}, X_{80}, X_{517}, X_{953}$		(e)*
X_{519}	$X_1, X_{80}, X_{519}, X_{2718}$		(5)
X_{528}	$X_1, X_{11}, X_{80}, X_{105}, X_{528}, X_{1156}$		
X_{3065}	$X_1, X_{35}, X_{80}, X_{191}, X_{484}, X_{502}, X_{3065}$		(f)*

Recall that the notations (x) and (x)* refer to a cubic (x) and its isogonal transform (x)*.

6.3 $\mathcal{C}(X_2, U)$ and $\mathcal{C}(X_6, U)$

Table 4: A selection of remarkable focals $\mathcal{C}(X_2, U)$ and $\mathcal{C}(X_6, U)$

P	U	centers on $\mathcal{C}(P, U)$	cubic	note
X_2	X_3	$X_2, X_3, X_{67}, X_{98}, X_{182}, X_{186}, X_{187}$	K300*	
X_2	X_4	$X_2, X_4, X_{30}, X_{67}, X_{98}, X_{316}, X_{1352}$	K477	(2)
X_2	X_{15}	$X_2, X_{14}, X_{15}, X_{111}, X_{187}$		
X_2	X_{16}	$X_2, X_{13}, X_{16}, X_{111}, X_{187}$		
X_2	X_{265}	$X_2, X_4, X_{67}, X_{265}, X_{316}, X_{542}$	K481	
X_2	X_{511}	$X_2, X_3, X_4, X_{67}, X_{187}, X_{316}, X_{511}, X_{842}$	K473	
X_2	X_{524}	$X_2, X_{13}, X_{14}, X_{67}, X_{524}, X_{2770}$	K065	(5)
X_2	X_{843}	$X_2, X_{15}, X_{16}, X_{187}, X_{352}, X_{524}, X_{843}$	K435	(5)
X_6	X_{13}	$X_6, X_{13}, X_{16}, X_{524}, X_{671}$		
X_6	X_{14}	$X_6, X_{14}, X_{15}, X_{524}, X_{671}$		
X_6	X_{25}	$X_6, X_{23}, X_{25}, X_{895}, X_{1995}$		
X_6	X_{111}	$X_6, X_{15}, X_{16}, X_{23}, X_{111}, X_{2854}$		
X_6	X_{524}	$X_6, X_{524}, X_{671}, X_{843}$		(5)
X_6	X_{543}	$X_6, X_{13}, X_{14}, X_{111}, X_{543}, X_{671}$	K063	

6.4 $\mathcal{C}(X_3, U)$ and $\mathcal{C}(X_4, U)$

Table 5: A selection of remarkable focals $\mathcal{C}(X_3, U)$ and $\mathcal{C}(X_4, U)$

P	U	centers on $\mathcal{C}(P, U)$	cubic	note
X_3	X_5	$X_3, X_5, X_{54}, X_{186}, X_{252}, X_{1141}, X_{1157}$	K467	
X_3	X_6	$X_3, X_6, X_{23}, X_{74}, X_{378}, X_{511}$		(3)
X_3	X_{74}	X_3, X_{74}, X_{186}	iK187	
X_3	X_{98}	$X_2, X_3, X_{98}, X_{186}, X_{542}, X_{1691}$	iK433	(1)
X_3	X_{671}	$X_3, X_4, X_6, X_{23}, X_{98}, X_{542}, X_{671}, X_{2080}$	K302	(4)
X_3	X_{842}	$X_3, X_{23}, X_{186}, X_{187}, X_{249}, X_{511}, X_{842}$	iK072	
X_3	X_{1263}	$X_3, X_4, X_{54}, X_{1141}, X_{1157}, X_{1263}, X_{2070}$	K466	
X_3	X_{2698}	$X_3, X_{186}, X_{237}, X_{2698}$	iK166	
X_3	E_{487}	X_3, X_{25}, X_{186}	iK164	
X_4	X_5	$X_4, X_5, X_{30}, X_{1141}$		(2)
X_4	X_6	$X_4, X_6, X_{23}, X_{262}, X_{265}, X_{381}, X_{511}, X_{576}, X_{671}$	K300	(4)
X_4	X_{15}	$X_4, X_{14}, X_{15}, X_{62}, X_{511}, X_{621}, X_{1337}$		
X_4	X_{16}	$X_4, X_{13}, X_{16}, X_{61}, X_{511}, X_{622}, X_{1338}$		
X_4	X_{23}	$X_4, X_6, X_{23}, X_{30}, X_{671}, X_{842}$	K298	(2)(4)
X_4	X_{30}	$X_4, X_{30}, X_{265}, X_{477}$	K530	(5)
X_4	X_{54}	$X_4, X_5, X_{54}, X_{143}, X_{195}, X_{265}, X_{1154}, X_{1263}, X_{2070}$	K464	
X_4	X_{511}	$X_4, X_6, X_{265}, X_{316}, X_{511}, X_{842}, X_{1916}$		
X_4	X_{542}	$X_4, X_{67}, X_{98}, X_{115}, X_{265}, X_{542}, X_{671}$	K301	(4)
X_4	X_{1157}	$X_3, X_4, X_5, X_{1154}, X_{1157}, X_{1263}$	K465	(5)

(1) $\mathcal{C}(X_3, X_{98})$ is also $\mathcal{C}(X_2, X_{316}^*)$ where X_{316}^* is the isogonal conjugate of the Droussent pivot.

6.5 Other cubics

Table 6: Other remarkable focals $\mathcal{C}(P, U)$

P	U	centers on $\mathcal{C}(P, U)$	cubic	note
X_{13}	X_{16}	$X_{13}, X_{14}, X_{15}, X_{16}, X_{530}, X_{2379}$	K261b	
X_{14}	X_{15}	$X_{13}, X_{14}, X_{15}, X_{16}, X_{531}, X_{2378}$	K261a	
X_{13}	X_{62}	$X_4, X_{13}, X_{16}, X_{62}, X_{628}, X_{1338}$		(1)
X_{14}	X_{61}	$X_4, X_{14}, X_{15}, X_{61}, X_{627}, X_{1337}$		(1)

(1) $\mathcal{C}(X_{13}, X_{62}) = \mathcal{C}(X_4, X_{621}^*)$, $\mathcal{C}(X_{14}, X_{61}) = \mathcal{C}(X_4, X_{622}^*)$.

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