Circumcubics meeting the Circumcircle at six points with Concurring Tangents

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Abstract

We study the cubics meeting the circumcircle of a triangle at six points with concurring tangents and also several remarkable related conics.

In [6], §10.4, we have met a class of cubics meeting the circumcircle $(\mathcal{O})$ of the reference triangle ABC at six points with tangents concurring at the Lemoine point K. The polar conic of K in these cubics is therefore $(\mathcal{O})$. The satellite conic[1] of $(\mathcal{O})$ is always the Brocard ellipse.

The goal of this paper is to generalize these facts and identify all the circumcubics having analogous properties.

1 Theorems and definitions

1.1 Generalities

Let $\mathcal{K}$ be a circumcubic i.e. a cubic passing through the vertices of the reference triangle ABC. $\mathcal{K}$ meets the circumcircle $(\mathcal{O})$ of ABC again at three other points Q_1, Q_2, Q_3 and one of these points is always real.

We are interested in such cubics whose tangents at these six points are concurrent at a given point $P = u : v : w$. The circumcevian triangle of P is hereby denoted by $P_aP_bP_c$ where $P_a = -a^2vw : v(c^2v + b^2w) : w(c^2v + b^2w)$, P_b, P_c likewise.

The conic $\Gamma_0(P)$ inscribed in both triangles ABC, $P_aP_bP_c$ is that with perspector Z_0 with first barycentric coordinate :

$$\frac{u}{a^2(b^2w + c^2v)}.$$

The first and immediate consequence of this property is that the polar conic $\mathcal{C}(P)$ of P with respect to $\mathcal{K}$ is $(\mathcal{O})$. It follows that the polar line $\mathcal{L}(P)$ of P with respect to $\mathcal{K}$ is the polar line of P with respect to $(\mathcal{O})$.

$\mathcal{L}(P)$ is the trilinear polar of the cevapoint Z_2 of K and P. The equation of $\mathcal{L}(P)$ and the coordinates of Z_2 are respectively :

$$\sum(c^2v + b^2w)x = 0, \quad Z_2 = \frac{1}{c^2v + b^2w} : \frac{1}{a^2w + c^2u} : \frac{1}{b^2u + a^2v}. \tag{1}$$

Remark that $\mathcal{L}(P)$ is also the axis of the perspectivity (homology) with center P that swaps the triangles ABC and $P_aP_bP_c$. Thus, $\mathcal{L}(P)$ contains $BC \cap P_bP_c$, $CA \cap P_cP_a$ and $AB \cap P_aP_b$.

[1] A line meets a cubic at three points. The tangentials of these three points lie on a same line called the satellite line of the former line. Similarly, a conic meets a cubic at six points. The tangentials of these six points lie on a same conic called the satellite conic of the former conic.

Let us recall that if a variable line passes through P and meets $\mathcal{K}$ at M_1, M_2, M_3 then $\mathcal{C}(P)$ is the locus of M on this line defined by

$$\frac{\overline{PM_1}}{\overline{MM_1}}+\frac{\overline{PM_2}}{\overline{MM_2}}+\frac{\overline{PM_3}}{\overline{MM_3}}=0 \iff \frac{3}{\overline{MP}}=\frac{1}{\overline{MM_1}}+\frac{1}{\overline{MM_2}}+\frac{1}{\overline{MM_3}} \tag{2}$$

where $\overline{MN}$ denotes the signed distance from M to N. In other words, P must be the center of mean distances from M to the points M_1, M_2, M_3 i.e. $\overline{MP}$ is the harmonic mean of the distances $\overline{MM_i}$.

The line AP meets $\mathcal{K}$ at A (twice) and at A', the tangential of A in the cubic. Since P_a lies on $\mathcal{C}(P)=(\mathcal{O})$ equation (2) gives :

$$\frac{3}{\overline{P_aP}}=\frac{2}{\overline{P_aA}}+\frac{1}{\overline{P_aA'}} \tag{3}$$

This same line AP meets $\mathcal{L}(P)$ at A'' which is the harmonic conjugate of P with respect to A and P_a. It follows that :

$$\frac{2}{\overline{P_aA}}=\frac{1}{\overline{P_aP}}+\frac{1}{\overline{P_aA''}} \tag{4}$$

The combination of equations (3) and (4) gives :

$$\frac{2}{\overline{P_aP}}=\frac{1}{\overline{P_aA'}}+\frac{1}{\overline{P_aA''}} \tag{5}$$

which shows that A' is the harmonic conjugate of A'' with respect to P and P_a i.e A' is the point such that $(P, P_a, A'', A') = -1$.

It follows that any circumcubic with six tangents concurring at P must pass through three other known points A', B', C' (the tangentials of A, B, C in the cubic) which are independent of the position of Q_1, Q_2, Q_3 on $(\mathcal{O})$.

The barycentric coordinates of A' are : $u-\dfrac{2a^2vw}{c^2v+b^2w}:3v:3w$, those of B' and C' likewise.

Note that this triangle $A'B'C'$ is perspective with the tangential triangle (i.e. the anticevian triangle of K) and obviously with the anticevian triangle of P. More generally, it is perspective with the anticevian triangle of any point on the pivotal cubic with pivot Z_2 and isopivot the perspector of the circum-conic $\Gamma(P)$ we meet in §2.4.

1.2 Theorem

An immediate consequence of the previous paragraph is

Theorem 1. *In general, for a given point P, all the circumcubics meeting the circumcircle $(\mathcal{O})$ of ABC at six points with tangents passing through P form a pencil $\mathcal{F}_P$ of cubics.*

Obviously, this pencil contains the degenerate cubic which is the union of the cevian lines of P.

When we apply the same reasoning to the points $Q_i, i \in \{1,2,3\}$, we find that $(P, P_i, Q''_i, Q'_i) = -1$ where P_i is the second intersection of the line PQ_i with $(\mathcal{O})$, Q'_i is the tangential of Q_i in the cubic and Q''_i is the intersection of PQ_i and $\mathcal{L}(P)$.

1.3 The satellite conic $\mathcal{S}(P)$

Hence, the satellite conic $\mathcal{S}(P)$ of $(\mathcal{O})$ with respect to the cubic is the image of $(\mathcal{O})$ under the mapping f_P that transforms any point M on $(\mathcal{O})$ into the point M' such that $(P, N, M'', M') = -1$. Here N is the second intersection of the line PM with $(\mathcal{O})$ and M'' is the intersection of the lines PM and $\mathcal{L}(P)$. Note that the harmonic conjugate N' of M' with respect to P and M'' also lies on $\mathcal{S}(P)$ and we have $(P, M, M'', N') = -1$ or equivalently $(P, M'', M', N') = -1$. This yields $(M, N, M', N') = 9$ and $(M, M', N, N') = -8$.

To realize the construction of $\mathcal{S}(P)$, it is sufficient to take M successively at A, B, C in order to obtain six points on the conic. In particular, $\mathcal{S}(P)$ contains A' and the harmonic conjugate of A'' with respect to P and A. Four other points are defined similarly.

With $M = x : y : z$ on $(\mathcal{O})$, the first barycentric coordinate of $M' = f_P(M)$ is :

$$\left(\sum a^2\, u\, y^2 z^2\right) u + 2a^2\, yz\, (vx - uy)(uz - wx),$$

the other coordinates similarly.

From this, we can easily obtain the equation of $\mathcal{S}(P)$:

$$3\left(\sum (c^2 v + b^2 w) x\right)^2 - 16\left(\sum a^2 vw\right)\left(\sum a^2 yz\right) = 0 \qquad (6)$$

This equation shows that $\mathcal{S}(P)$ is bitangent to $(\mathcal{O})$ at the points T_1, T_2 where the polar line $\mathcal{L}(P)$ of P with respect to $(\mathcal{O})$ meets $(\mathcal{O})$. These points are real when P is outside $(\mathcal{O})$ and they coincide when P lies on $(\mathcal{O})$. This property also derives of the definition of f_P. Indeed, when the line PM is tangent to $(\mathcal{O})$, we must have $M = N = M''$ hence $M' = M$. Note that $\mathcal{L}(P)$ is also the polar line of P in $\mathcal{K}$ and in $\mathcal{S}(P)$. See figure 1.

Remarks :

1. $\mathcal{S}(P)$ is a circle if and only if $P = O$ and then $\mathcal{S}(P) = \mathcal{C}(O, R/2)$. See §3 below.
2. when $P \neq O$, the center of $\mathcal{S}(P)$ lies on the line OP.
3. $\mathcal{S}(P)$ decomposes (into a double line) if and only if P lies on $(\mathcal{O})$. See §2.3.
4. $\mathcal{S}(P)$ is a parabola if and only if P lies on the circle $\mathcal{C}(O, 2R)$.
5. $\mathcal{S}(P)$ is a rectangular hyperbola if and only if P lies on the circle $\mathcal{C}(O, \sqrt{\frac{8}{5}}R)$.

2 The cubics $\mathcal{K}(P, \lambda)$

2.1 Definition

A computation shows that any cubic $\mathcal{K}(P, \lambda)$ of $\mathcal{F}_P$ has an equation of the form :

$$\begin{aligned} &uvw\left[\sum (c^2 v - b^2 w)\, x^2 (wy - vz) + 2\left(\sum a^2 vw\right) xyz\right] \\ &\quad + 2\lambda\left(\sum a^2 vw\right)(wy - vz)(uz - wx)(vx - uy) = 0 \end{aligned} \qquad (7)$$

where λ is any number in $\mathbb{R} \cup \{\infty\}$.

This is actually the equation of a *ps*$\mathcal{K}$ as in [3]. Further details in §2.4.

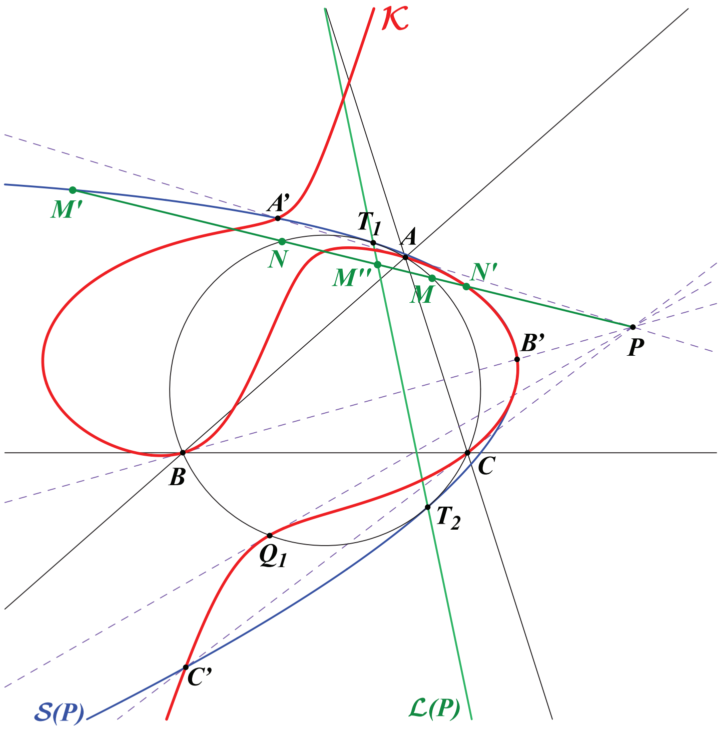

Figure 1: $\mathcal{K}$, $\mathcal{S}(P)$ and $\mathcal{L}(P)$

Assuming that P is not on $(\mathcal{O})$ (see §2.3 to cover the case P on $(\mathcal{O})$), when $\lambda = \infty$ we obtain the union of the cevian lines of P and when $\lambda = 0$ we obtain another cubic $\mathcal{K}'$ of the pencil $\mathcal{F}_P$ with equation :

$$\sum (c^2v - b^2w)x^2(wy - vz) + 2\left(\sum a^2vw\right) xyz = 0. \tag{8}$$

$\mathcal{K}'$ meets the sidelines of ABC at three points U, V, W which are the vertices of the cevian triangle of S, the trilinear pole of the line passing through K and P (see below for the case $P = K$).

$S = \dfrac{1}{c^2v - b^2w} : \dfrac{1}{a^2w - c^2u} : \dfrac{1}{b^2u - a^2v}$ is clearly a point on $(\mathcal{O})$.

Note that the cevian triangle UVW of S is self-polar in $(\mathcal{O})$. See figure 2.

Naturally, these points U, V, W lie on the pivotal cubic with pivot S, isopivot P which is also tangent at A, B, C to the cevian lines of P and at S to the line PS.

2.2 Special case : $P = K$

When $P = K$ (Lemoine point), the equation (7) considerably simplifies. Any cubic $\mathcal{K}(K, \lambda)$ of $\mathcal{F}_K$ can be written under the form

$$\lambda(c^2y - b^2z)(a^2z - c^2x)(b^2x - a^2y) + a^2b^2c^2xyz = 0, \tag{9}$$

from which we see that $\mathcal{K}(K, \lambda)$ is a $ps\mathcal{K}$ with pseudo-pole X_{32} and pseudo-pivot $K = X_6$.

The satellite conic $\mathcal{S}(K)$ of $(\mathcal{O})$ is the Brocard ellipse for any λ.

This is the only case where the points A', B', C' are the vertices of a cevian triangle, that of K. These cubics are precisely those we had met in [6] (see the remark in §10.4). See figure 3.

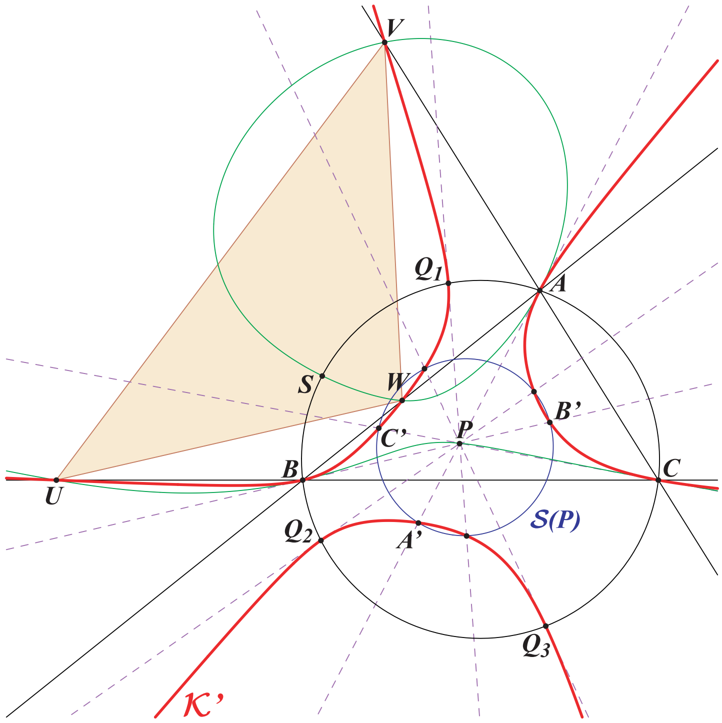

Figure 2: The cubic $\mathcal{K}'$

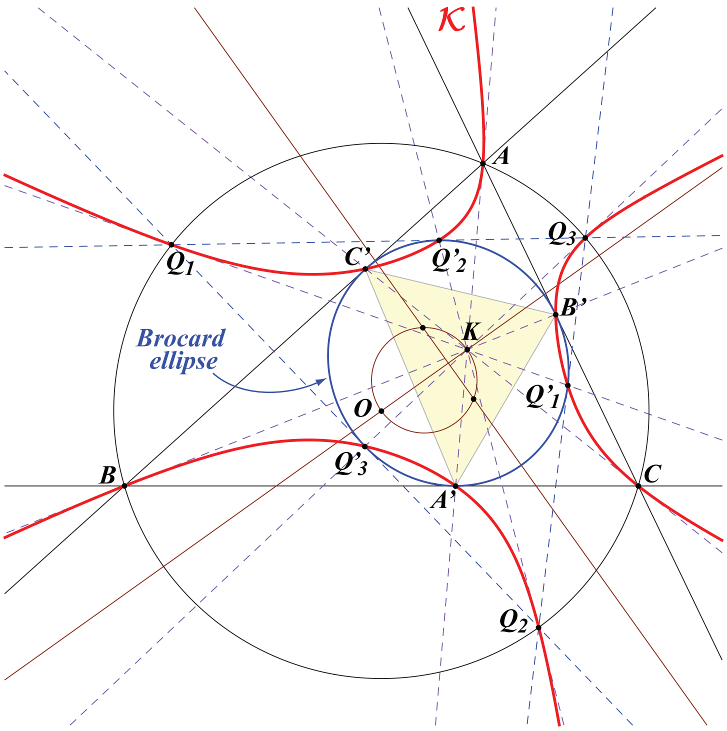

Figure 3: A cubic $\mathcal{K}$ when $P = K$

For any point $Q = u : v : w$, there is a unique cubic of $\mathcal{F}_K$ passing through Q. This has the symmetrical equation

$$uvw \prod_{\text{cyclic}} (c^2y - b^2z) = xyz \prod_{\text{cyclic}} (c^2v - b^2w).$$

In particular, with $Q = H$, we obtain **K444** $= ps\mathcal{K}(X_{32}, X_6, X_4)$ passing through X_4, X_{32}, X_{194}, the Brocard points, the vertices of the third Brocard triangle, the infinite points of $p\mathcal{K}(X_6, X_{76})$. Note that Q_1, Q_2, Q_3 are also the common points of $(\mathcal{O})$ and **K410** $= p\mathcal{K}(X_6, X_{194})$. See figure 4.

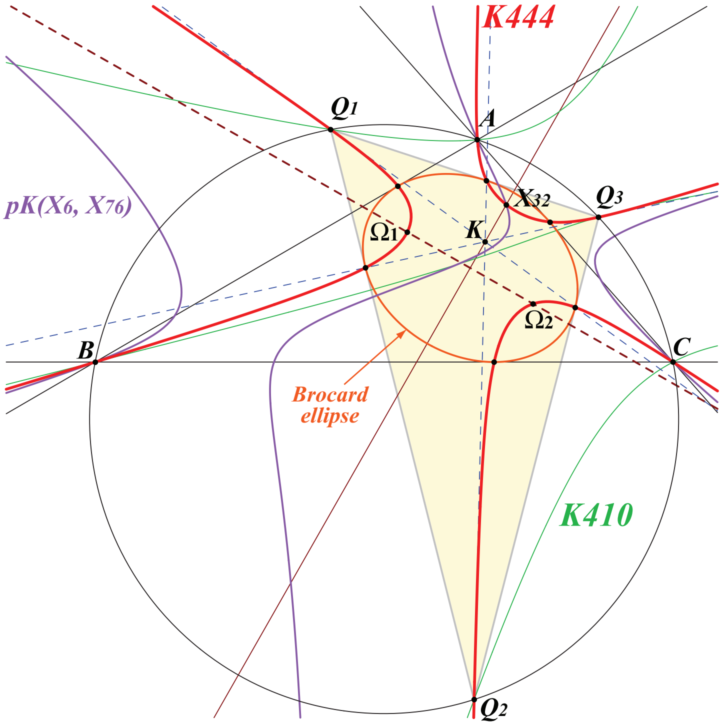

Figure 4: The cubic **K444** $= ps\mathcal{K}(X_{32}, X_6, X_4)$

Other examples are the axial cubic **K579** $= ps\mathcal{K}(X_{32}, X_6, X_{512})$ passing through X_{249}, X_{512} and also **K693** $= ps\mathcal{K}(X_{32}, X_6, X_2)$ passing through X_2, X_{184}, X_{3168}. See [2] for further details.

2.3 Special case : P on $(\mathcal{O})$

When P lies on $(\mathcal{O})$, equation (7) becomes

$$\sum (c^2v - b^2w)\, x^2(wy - vz) = 0 \tag{10}$$

showing that the cubic is a pivotal cubic with pivot S, the trilinear pole of the line KP, and isopivot P. These two points lie on $(\mathcal{O})$ and the cubic is tangent at P to $(\mathcal{O})$ and at S to the line PS. In this case, the satellite conic $\mathcal{S}(K)$ of $(\mathcal{O})$ degenerates into the tangent at P to $(\mathcal{O})$ counted twice.

Another thing to note is that this cubic is also a pivotal cubic with respect to the cevian triangle UVW of S. The pivot is now P and the isopivot P', the $S-$Ceva conjugate of P in the triangle ABC and also the tangential of P in $\mathcal{K}$. Hence, P' is the intersection of the tangent at P to $(\mathcal{O})$ and the trilinear polar of P. See figure 5.

Remark : when P traverses $(\mathcal{O})$, this point P' traverses the cubic **K229** $= n\mathcal{K}(X_{32}, X_6, X_6)$, an unicursal cubic with singular point K.

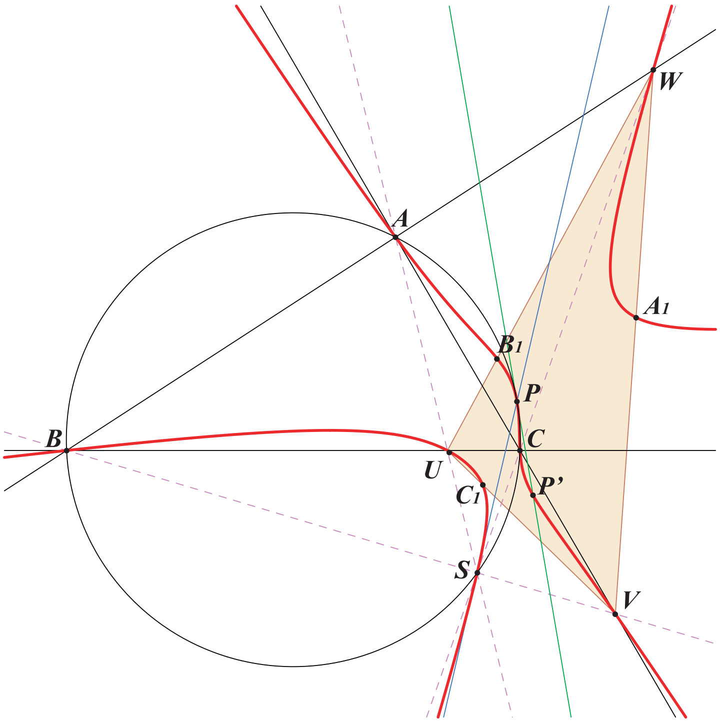

Figure 5: The cubic $\mathcal{K}$ when P is on $(\mathcal{O})$

2.4 The general case

From now on we suppose that $\mathcal{K} = \mathcal{K}(P, \lambda)$ is a cubic of $\mathcal{F}_P$ where P is not the Lemoine point K and does not lie on $(\mathcal{O})$. Recall that $\mathcal{K}$ is a *ps*$\mathcal{K}$ with respect to both triangles ABC and $Q_1Q_2Q_3$ whose pseudo-isopivot is P since its tangents at A, B, C and Q_1, Q_2, Q_3 concur at P.

$\mathcal{K}$ meets the sidelines of ABC at three other points U, V, W which are the vertices of a cevian triangle whose perspector is the point Q_λ, pseudo-pivot of the cubic, with coordinates

$$\frac{u}{u(c^2v - b^2w) + 2\lambda\,\Phi} : \frac{v}{v(a^2w - c^2u) + 2\lambda\,\Phi} : \frac{w}{w(b^2u - a^2v) + 2\lambda\,\Phi},$$

where $\Phi = \sum a^2vw$, clearly zero when P lies on $(\mathcal{O})$.

Q_λ lies on the circum-conic $\Gamma(P)$ passing through P and the trilinear pole S of the line KP. Indeed, the isotomic conjugate $\mathbf{t}Q_\lambda$ clearly lies on the line passing through $\mathbf{t}P$ and $\mathbf{t}S$. Note that S lies on $(\mathcal{O})$ and that the perspector Z_P of $\Gamma(P)$ lies on the line KP. The first coordinate of Z_P is $u(c^2uv + b^2uw - 2a^2vw)$ hence Z_P is the barycentric product of P and the infinite point of the line GP^*.

The pseudo-pole Ω_λ is therefore the barycentric product $P \times Q_\lambda$, a point on the circum-conic passing through P^2 (barycentric square) and $S \times P$.

It follows that any point Q_λ on $\Gamma(P)$ is associated with one and only one cubic $\mathcal{K}$ of $\mathcal{F}_P$ and vice versa. If $Q_\lambda = P$ (for $\lambda = \infty$), we obtain the cubic that is the union of the cevian lines of P and if $Q_\lambda = S$ (for $\lambda = 0$), we obtain the cubic $\mathcal{K}'$ above. See figure 6.

Remark : if Q_1 is given on $(\mathcal{O})$, the line Q_2Q_3 is the trilinear polar of the image of Q_1 under the isoconjugation that swaps Q_λ and the cevapoint Z_2 of K and P.

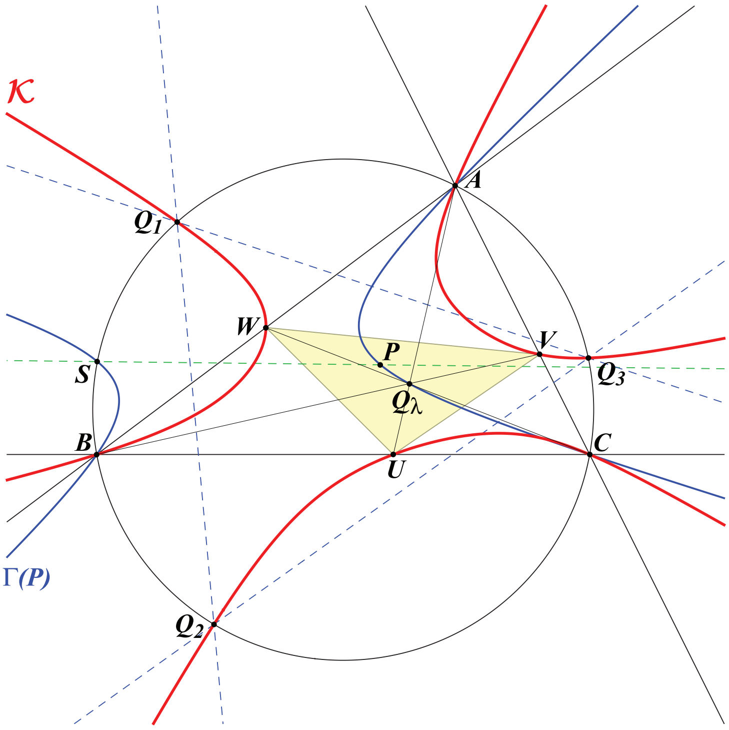

Figure 6: The cubic $\mathcal{K}$ and the conic $\Gamma(P)$

2.5 Points on $\mathcal{K}$

Apart the points A, B, C, their tangentials A', B', C' and the points U, V, W above, the cubic $\mathcal{K}$ contains the following other points :

- $A_1 = UA' \cap VW$, B_1 and C_1 likewise,
- $A_2 = BC_1 \cap CB_1$, B_2 and C_2 likewise,
- $A_3 = UA \cap A_1A_2$, B_3 and C_3 likewise,
- $U' = B'C' \cap AA_2$ (the tangential of U), V' and W' likewise.

We note that :
– the points A', B_3, C_3 are collinear,
– the points of each following sextuple all lie on a same conic :
$\{A, B, C, A', A_1, A_2\}$; $\{A, B, C, B', C', A_3\}$; $\{A', B', C', B, C, A_1\}$.

2.6 Other properties of $\mathcal{K}$

Most of the following results are obtained through calculation involving symmetric functions of the roots of a third degree polynomial although some of them have a synthetic projective explanation.

Proposition 1. *For a given P, all the cubics of $\mathcal{F}_P$ have their asymptotes forming a triangle whose Lemoine point is P.*

This is an immediate consequence of the fact that the polar conic of P is a circle. See [9].

Figure 7 also shows the poloconic of the line at infinity which is the inscribed Steiner ellipse of the triangle formed by the asymptotes. When this ellipse is not a circle (i.e. when the cubic

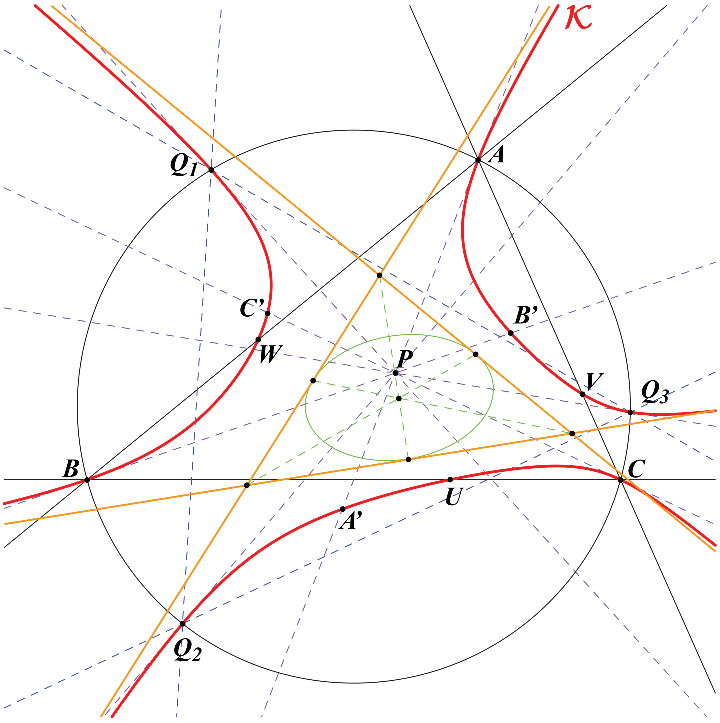

Figure 7: The cubic $\mathcal{K}$ and its asymptotes

is not equilateral), the polar of P with respect to this ellipse is the orthic axis of the triangle also called the orthic line of the cubic. It is the locus of points whose polar conic in the cubic is a rectangular hyperbola.

Proposition 2. *For a given P, all the cubics of $\mathcal{F}_P$ meet $(\mathcal{O})$ at three points Q_1, Q_2, Q_3 such that the conic $\mathcal{E}(P, \lambda)$ inscribed in both triangles ABC and $Q_1Q_2Q_3$ is tangent to a fixed line $\mathcal{T}(P)$ hence centered on another fixed line $\mathcal{N}(P)$.*

$\mathcal{N}(P)$ is the Newton line of the quadrilateral formed by the sidelines of ABC and $\mathcal{T}(P)$. See figure 8.

The first coordinate of the trilinear pole of $\mathcal{T}(P)$ is :

$$\frac{u(-2a^2vw + b^2uw + c^2uv)}{a^2(b^2w + c^2v)}$$

showing it is the barycentric product of Z_0 and the infinite point of the line GP^*.

Proposition 3. *For a given P, all the cubics of $\mathcal{F}_P$ meet $(\mathcal{O})$ at three points Q_1, Q_2, Q_3 such that the sidelines of $Q_1Q_2Q_3$ envelope a fixed conic $\mathcal{D}(P)$ tangent to $\mathcal{T}(P)$ and inscribed in the circumcevian triangle $P_aP_bP_c$ of P.*

See figure 9. The construction of $\mathcal{D}(P)$ follows from the knowledge of its perspector Z_P. First, we construct the barycentric product $Z_1 \times Z_2$ of the points Z_1 and Z_2 where

– Z_1 is the isotomic conjugate of the complement of the isogonal conjugate of P or, equivalently, the cevapoint of G and the isogonal conjugate P^* of P and also the trilinear pole of the polar line of P^* in the Steiner ellipse.

– Z_2 is the cevian product of K and P or, equivalently, the trilinear pole of $\mathcal{L}(P)$.

Then, we transform $Z_1 \times Z_2$ into Z_P under the homology h_P with center P, axis $\mathcal{L}(P)$ swapping A, B, C and P_a, P_b, P_c respectively. Note that h_P is an involution that globally fixes

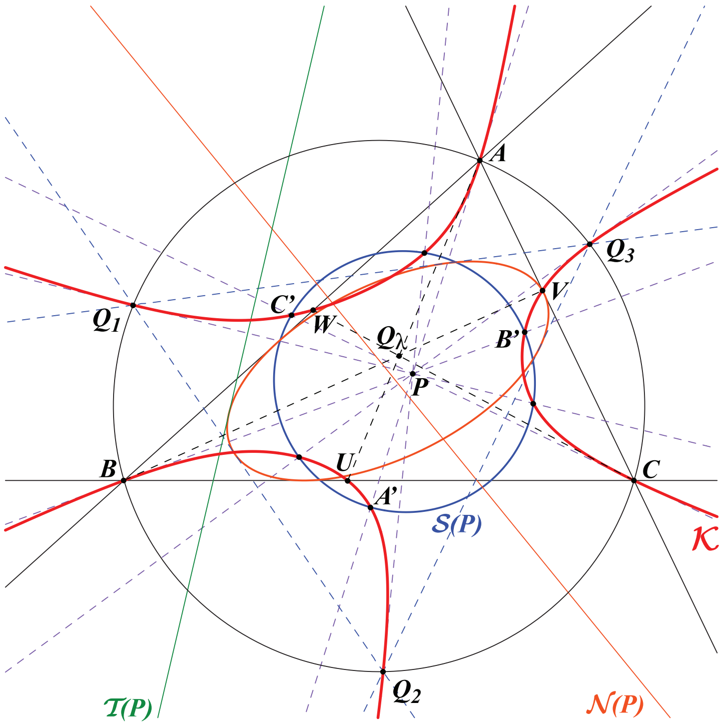

Figure 8: The cubic $\mathcal{K}$, $\mathcal{T}(P)$, $\mathcal{S}(P)$ and $\mathcal{N}(P)$

$(\mathcal{O})$. At last, draw the conic $\mathcal{D}(P)$ as the inscribed conic in $P_aP_bP_c$ with perspector Z_P. In other words, $\mathcal{D}(P)$ is the image under h_P of the conic inscribed in ABC whose perspector is $Z_1 \times Z_2$.

This same involution h_P gives $\mathcal{T}(P)$ as the image of the trilinear polar of the barycentric product $Z_2 \times Z_3$ where Z_3 is the infinite point of the trilinear polar of Z_1. This point $Z_2 \times Z_3$ obviously lies on $\mathcal{L}(P)$.

The first barycentric coordinates of Z_1, Z_2 and Z_3 are respectively :

$$\frac{1}{u(c^2v + b^2w)}, \quad \frac{1}{c^2v + b^2w}, \quad u(c^2v - b^2w).$$

When we fix Q_1 on $(\mathcal{O})$, we obtain one and only one cubic $\mathcal{K}$ passing through Q_1 and the construction of Q_2, Q_3 follows the construction of $\mathcal{D}(P)$. It is sufficient to draw the two tangents from Q_1 to $\mathcal{D}(P)$ meeting $(\mathcal{O})$ again at the required points Q_2, Q_3.

Proposition 4. *For a given P and a given Q_λ on $\Gamma(P)$ – or a given point Q_1 on $(\mathcal{O})$ – the cubic $\mathcal{K}$ meets the sidelines of $Q_1Q_2Q_3$ at three points R_1, R_2, R_3 such that*

1. *the triangles $Q_1Q_2Q_3$ and $R_1R_2R_3$ are perspective at Q_λ,*
2. *the points A', B', C', R_1, R_2, R_3 lie on a same conic $\Gamma_1(P)$,*
3. *the points U, V, W, R_1, R_2, R_3 lie on a same conic $\Gamma_2(P)$.*

The first assertion is a immediate consequence of the fact that $\mathcal{K}$ is a $ps\mathcal{K}$ in both triangles ABC and $Q_1Q_2Q_3$.

Recall that Q_λ is the perspector of the two triangles ABC and UVW. See figures 10 and 11.

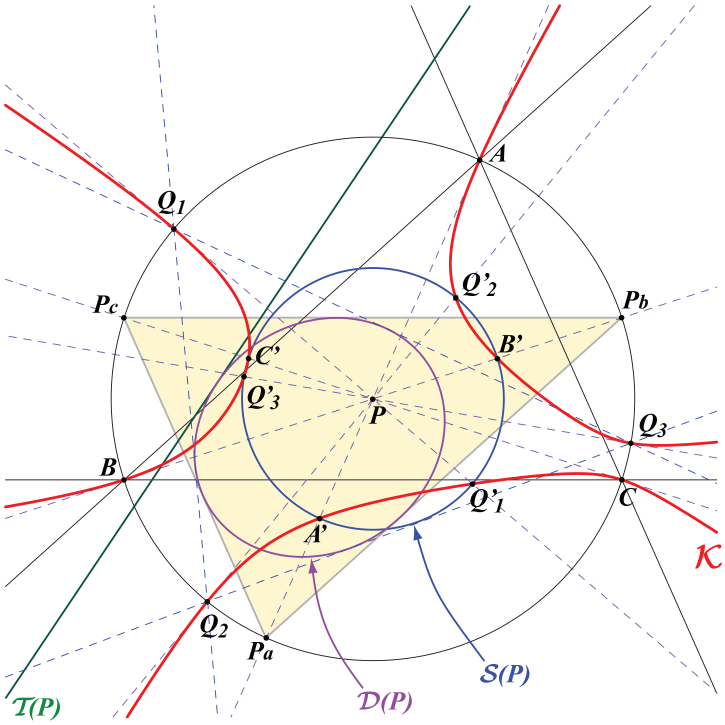

Figure 9: The cubic $\mathcal{K}$, $\mathcal{T}(P)$, $\mathcal{S}(P)$ and $\mathcal{D}(P)$ when $P = O$

Properties of these conics

1. $\Gamma_2(P)$ is a bicevian conic in ABC with perspectors Q_λ and $P_2 = Q_\lambda \times tgP$ where tgP is the isotomic conjugate of P^*.

2. It is also clear that these conics $\Gamma_1(P)$ and $\Gamma_2(P)$ are both bicevian conics in $Q_1Q_2Q_3$ with a common perspector Q_λ and two other perspectors $P_{1,\lambda}$, $P_{2,\lambda}$ respectively.

   $P_{1,\lambda}$ is actually Z_2 hence independent of λ. It follows that $\Gamma_1(P)$ can be seen as the mixed poloconic of the trilinear polars of Z_2 and Q_λ with respect to the decomposed cubic formed by the sidelines of $Q_1Q_2Q_3$.

   $P_{2,\lambda}$ is the barycentric quotient $agP \div Q_\lambda$ where agP is the anticomplement of P^*.

3. For a given point P and when λ varies, the conics $\Gamma_1(P)$ contain the (fixed) points A', B', C' and also a fourth fixed point R_0 which is the contact of $\mathcal{T}(P)$ with the conic $\Gamma_0(P)$ inscribed in both triangles ABC, $P_aP_bP_c$. The first coordinate of R_0 is :

$$\frac{u(-2a^2vw + b^2uw + c^2uv)^2}{a^2\,(b^2w + c^2v)}$$

   and recall that the first coordinate of the perspector Z_0 of $\Gamma_0(P)$ is :

$$\frac{u}{a^2(b^2w + c^2v)}$$

4. $\Gamma_1(P)$ and $\Gamma_2(P)$ contain R_1, R_2, R_3 and a fourth point R_λ which lies on $\mathcal{T}(P)$. This point R_λ is the second intersection (apart R_0) of $\mathcal{T}(P)$ and $\Gamma_1(P)$.

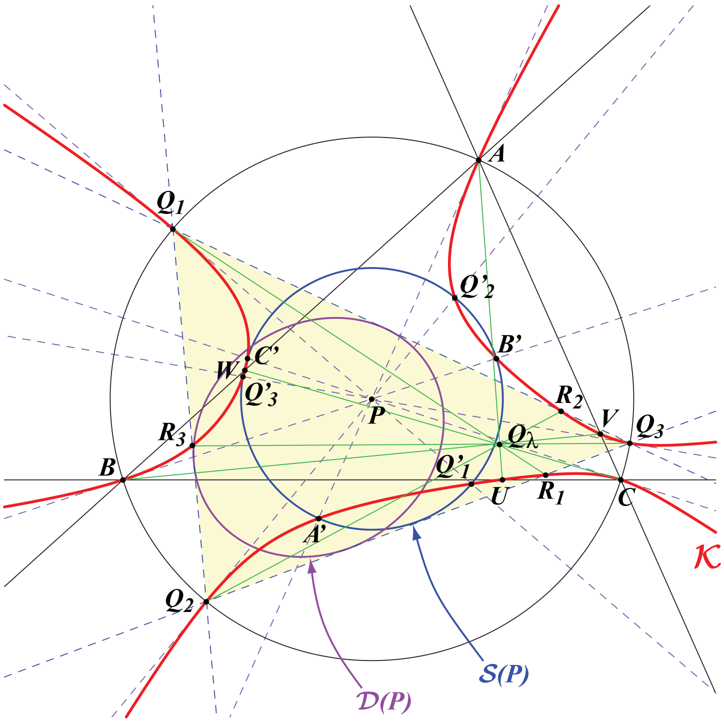

Figure 10: The cubic $\mathcal{K}$ and the three points R_1, R_2, R_3

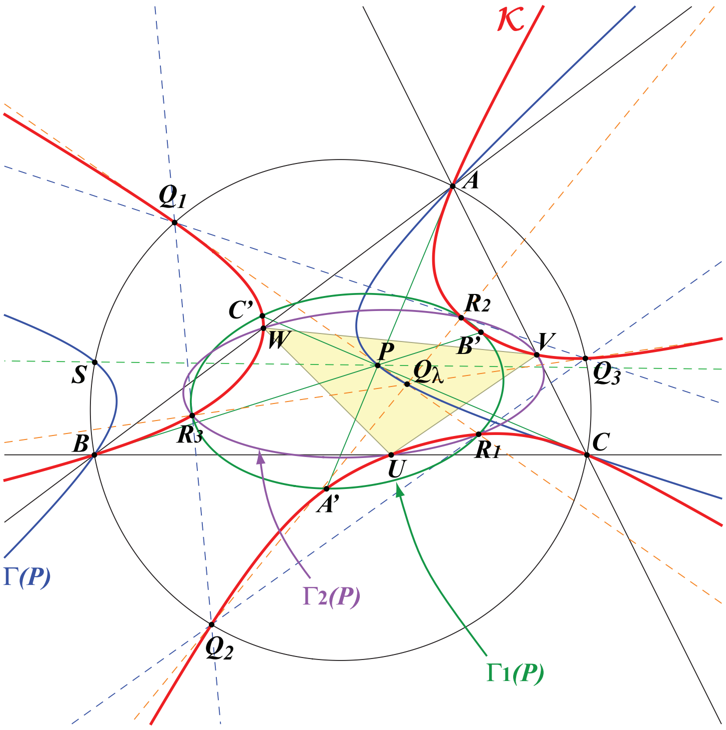

Figure 11: The cubic $\mathcal{K}$ and the conics $\Gamma_1(P)$, $\Gamma_2(P)$

3 Selected examples

We conclude this note with several remarkable examples of cubics $\mathcal{K}$.

3.1 K694 with $P = X_{512}$

This remarkable cubic meets $(\mathcal{O})$ at the vertices of an equilateral triangle $Q_1Q_2Q_3$ and the six tangents are perpendicular to the Brocard axis. The satellite conic $\mathcal{S}(P)$ is bitangent to $(\mathcal{O})$ at its intersections X_{1379}, X_{1380} with the Brocard axis. The center of $\mathcal{S}(P)$ is O. See figure 12. The pseudo-pivot of **K694** is the isogonal conjugate of the homothetic of X_{115} under $h(K, -2)$.

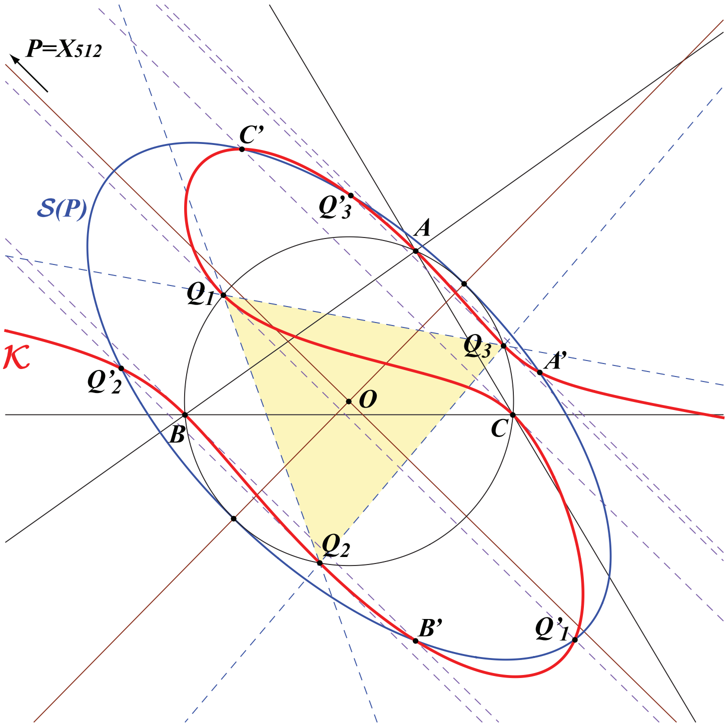

Figure 12: The special cubic **K694** obtained with $P = X_{512}$

More generally, the center of $\mathcal{S}(P)$ is O if and only if P lies on the line at infinity or $P = O$, see below.

In the former case, $\mathcal{S}(P)$ is an ellipse tangent to the circle $\mathcal{C}(O, 2R)$ and to the circumcircle, the contacts being on the line OP and its perpendicular at O respectively.

3.2 $P = O$

This is the only case when $\mathcal{S}(P)$ is a circle namely the circle with center O, radius half of that of $(\mathcal{O})$. The figure 13 shows one of these cubics, that passing through X_{74} and X_{1511} labelled Q_3 and Q'_3 on the figure.

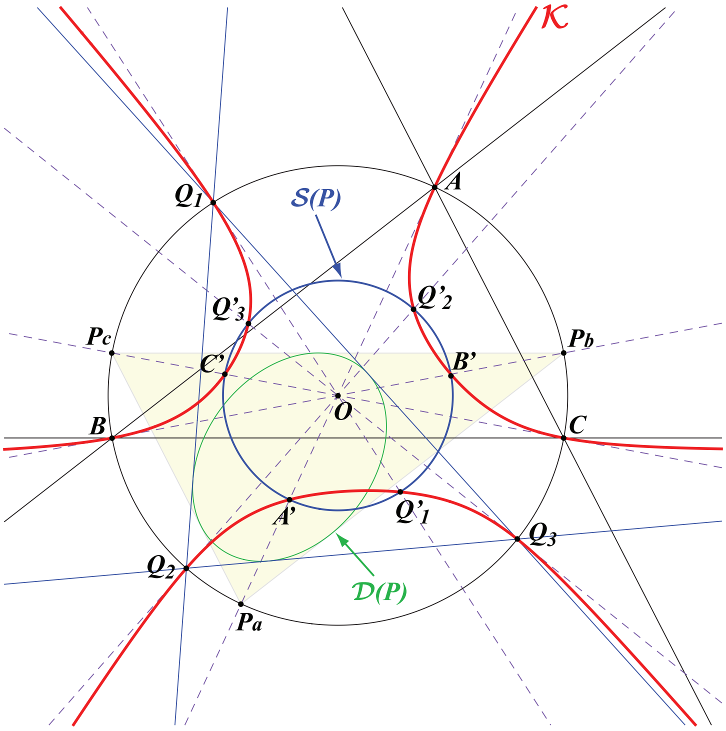

Figure 13: A cubic $\mathcal{K}$ when $P = O$

References

[1] J.P. Ehrmann and B. Gibert, *Special Isocubics in the Triangle Plane*, available at http://bernard.gibert.pagesperso-orange.fr

[2] B. Gibert, *Cubics in the Triangle Plane*, available at http://bernard.gibert.pagesperso-orange.fr

[3] B. Gibert, Pseudo-Pivotal Cubics and Poristic Triangles, available at http://bernard.gibert.pagesperso-orange.fr

[4] B. Gibert, *The Lemoine Cubic and its Generalizations*, Forum Geometricorum, vol.2, pp.47-63, 2002.

[5] B. Gibert, *How Pivotal Isocubics intersect the Circumcircle*, Forum Geometricorum, vol.7, pp.211-229, 2007.

[6] B. Gibert, *Cubics related with Coaxial Circles*, Forum Geometricorum, vol.8, pp.77-95, 2008.

[7] C. Kimberling, *Triangle Centers and Central Triangles*, Congressus Numerantium, 129 (1998) 1–295.

[8] C. Kimberling, *Encyclopedia of Triangle Centers*, 2000-2008 http://faculty.evansville.edu/ck6/encyclopedia/ETC.html

[9] Stuyvaert *Point remarquable dans le plan d'une cubique*, Nouvelles annales de mathématiques, Série. 3, 18 (1899), pp. 275–285.